COUNTING METHODS

FUNDAMENTAL COUNTING PRINCIPLE:

If an experiment consists of a series of successive steps, then the total number of outcomes for the experiment is determined by multiplying the number of outcomes for each step.

examples:

If a die is rolled and then a coin is tossed, find the number of possible outcomes.
Answer: (6)(2) = 12 (the 12 outcomes in the sample space are shown below)
S = { 1H, 1T, 2H, 2T, 3H, 3T, 4H, 4T, 5H, 5T, 6H, 6T }

A five place computer ID may use capital letters A to Z or the digits 0 to 9. If the first place must be a letter, and an ID may not use repetition, how many different IDs are possible?
Answer: (26)(35)(34)(33)(32) = 32,672,640

FACTORIAL NOTATION: If n is a positive integer, n! (read as "n factorial") is the product of all positive integers from n down through 1.

$n! = n(n-1)(n-2) \ldots (3)(2)(1)$

example: $5! = (5)(4)(3)(2)(1) = 120$ [note: by definition, $0! = 1$]

PERMUTATIONS:

The number of permutations (or arrangements) of n distinct items, taken r at a time is

$${}_nP_r = \frac{n!}{(n-r)!}$$

[note: order matters!]

example: From 7 people, how many different ways can 3 people be selected to serve as president, vice-president, and treasurer?

Answer: $${}_7P_3 = \frac{7!}{(7-3)!} = \frac{7!}{4!} = \frac{(7)(6)(5)(4)(3)(2)(1)}{(4)(3)(2)(1)} = (7)(6)(5) = 210$$

*(Note that use of the fundamental counting principle will produce the same answer)

COMBINATIONS:

The number of combinations (or subsets) of n distinct items, taken r at a time is

$${}_nC_r = \frac{n!}{(n-r)!\,r!}$$

[note: order does not matter!]

example: From 7 people, how many different ways can 3 people be selected to form a committee?

Answer: $${}_7C_3 = \frac{7!}{(7-3)!\,3!} = \frac{7!}{4!\,3!} = \frac{(7)(6)(5)(4)(3)(2)(1)}{[(4)(3)(2)(1)]\,[(3)(2)(1)]} = 35$$

To avoid the common error of confusion between permutations and combinations, note again that in permutations order matters, but in combinations order does not matter.
In the permutations example, Al, Bob, and Sue as president, vice-president, and treasurer is a different result than having Sue, Bob, and Al in the respective offices.
However, a committee of Al, Bob, and Sue is the same as a committee of Sue, Bob, and Al.
Noting that combinations and committee both begin with the letter c may be helpful.

PROBABILITIES INVOLVING INTERSECTION (AND)

If we perform an experiment, we may want to know the probability that an outcome occurs that fits the description of two different events A and B. Or if an experiment is performed twice, we may want to know the probability that the events A and B occur sequentially. In both cases, this is referred to as the probability of A intersection B.
The word and is often used to represent intersection, so this probability is often represented as P(A and B).

$$P(A \text{ and } B) = P(A) \bullet P(B \text{ given that A has occurred})$$

As an example, let's use the set of five objects shown at the right: □ ● ▲ ○ ◆

If we randomly selected one of the above objects, what is the probability that the object is black and triangular?

Of course, note that that the simplest way to answer the question is to observe that of the five equally likely objects to be selected, only one is both black and triangular, thus P(B and T) = 1/5.
Consequently, it is best to use the formula only in the case of multiple step experiments, to be demonstrated later.

But also note that that the intersection formula (although requiring more work in this case) can be applied.
P(B and T) = P(B) • P(T given B) = 3/5 • 1/3 = 1/5

Suppose that we have a shipment of 7 parts (4 good and 3 defective) as shown at the right: D G G G G D D

If we randomly select two parts for inspection (naturally, without replacement),
what is the probability that the first part is good and the second part is defective?

In symbols, we are asking what is $P(G_1 \text{ and } D_2)$?
On the first selection 4 of the 7 parts are good, but on the second selection 3 of the remaining 6 parts are defective.
Therefore, $P(G_1 \text{ and } D_2) = P(G_1) \bullet P(D_2 \text{ given } G_1) = 4/7 \bullet 3/6 = 2/7$

Also observe that questions involving intersection don't always use the word and.
For example, we may ask what is the probability that both parts selected are defective?
The word and is not in the question, but both good means that the first is defective and the second is defective.
Therefore, this probability is $P(D_1 \text{ and } D_2) = P(D_1) \bullet P(D_2 \text{ given } D_1) = 3/7 \bullet 2/6 = 1/7$

We can determine the probability that 3 consecutive good parts are selected by intuitively extending the formula.
$P(G_1 \text{ and } G_2 \text{ and } G_3) = 4/7 \bullet 3/6 \bullet 2/5 = 4/35$

The last three questions involved **dependent events,** where since we are not replacing selected parts, the probability of selecting a good or defective part depends on which type of part was selected (and thus removed) previously.
Had we been replacing previously selected parts (which of course wouldn't have made sense in an inspection scenario) the events would have been independent. With **independent events,** we need not be concerned with what occurred on previous trials of an experiment since they will not influence what occurs on future trials.
Examples of independent events involve tosses of a coin, rolls of a die, or sampling with replacement.

$$\text{If events A and B are independent, then } P(A \text{ and } B) = P(A) \bullet P(B)$$

Some examples follow, and also note that this formula can similarly be extended to any number of independent events.

If a fair die is rolled, what is the probability that the number 5 appears on both rolls? (1/6)(1/6) = 1/36

If a fair coin is tossed three times, what is the probability that heads appears all three times? (1/2)(1/2)(1/2) = 1/8

If a four-place password consists strictly of the capital letters A to Z (with repetition allowed), what is the probability that a thief can guess the password on the first try? (1/26)(1/26)(1/26)(1/26) =1/456,976

If a fair coin is tossed six times, what is the probability that heads appears at least once?
(Here we will use the complement)
P(heads at least once) = 1 – P(heads no times)
$= 1 - P(\text{all tails}) = 1 - (1/2)(1/2)(1/2)(1/2)(1/2)(1/2) = 1 - (1/2)^6 = 1 - 1/64 = 63/64$

PROBABILITIES INVOLVING UNION (OR)

If we perform an experiment, we may want to know the probability that an outcome occurs that fits the description of event A, or event B, or both events. This is referred to as the probability of A union B. The word or is often used to represent union, so this probability is often represented as P(A or B).

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

As an example, let's use the set of five objects shown at the right: □ ● ▲ ○ ◆

If we randomly select one of the above objects, what is the probability that the object is black or circular?

In a case such as this, where we can easily see each of the equally likely outcomes, we can simply observe that 4 of the 5 objects (the last four) fit the description of black, or circular, or both. So the probability is 4/5.

However, let's try the formula since it works in all cases and will be needed in many cases.

$P(B \text{ or } C) = P(B) + P(C) - P(B \text{ and } C)$
$= 3/5 + 2/5 - 1/5$
$= 4/5$

Note that we must subtract P(B and C) in order to avoid double counting an object (in this case the black circle) that fits both descriptions.

Let's try an example now where the formula is needed.

Suppose that two shooters, Al and Bob, each take one shot at a target.
Also, suppose that Al can hit the target with a probability of 0.9 while Bob can do so with a probability of 0.8.
Assuming independence, what is the probability that at least one of the two shooters hits the target?

Understand that this question is just asking us to find P(A or B) since at least one of the two events A or B occurring means that A occurs, or B occurs, or both occur.

$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$
$= P(A) + P(B) - P(A)P(B)$ [by independence]
$= 0.9 + 0.8 - (0.9)(0.8)$
$= 1.7 - 0.72 = 0.98$

Mutually exclusive events are events that cannot occur at the same time.
Examples (pairs of mutually exclusive events) are shown below.

winning a game, losing a game
making an A on a test, making a C on a test

Consequently, if A and B are mutually exclusive events, then P(A and B) = 0.
Resultingly, the formula for the union of two mutually exclusive events simplifies to the sum of their probabilities.

If events A and B are **mutually exclusive**, then $P(A \text{ or } B) = P(A) + P(B)$

Let's apply the formula above for the simple case below.

If a fair die is rolled, what is the probability a 2 or a 5 is rolled?

Since the events are mutually exclusive (we can't roll a 2 and a 5 simultaneously),
$P(2 \text{ or } 5) = P(2) + P(5) = 1/6 + 1/6 = 2/6 = 1/3$

EXPECTED VALUE (EXPECTATION)

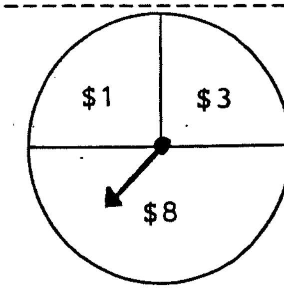
In a game of chance or business venture, what is usually of greatest concern is not any short term gain or loss, but the long term average (expected value or expectation) which is the average gain or loss resulting from many repetitions of an experiment (such as playing a game thousands of times or delivering thousands of packages).
Expected value is computed as: $\mathbf{E = A_1(P_1) + A_2(P_2) + \ldots + A_n(P_n)}$
Where $A_1, A_2, \ldots A_n$ are the amounts associated with the n outcomes of an experiment
and $P_1, P_2, \ldots P_n$ are the corresponding probabilities for the outcomes.
Note that we simply multiply each amount by its corresponding probability and add these results.
What we are calculating is a weighted average. If an amount is highly probable, it is multiplied by its high probability, thus giving it a high amount of weight or influence. Contrastingly, a less probable amount is multiplied by its smaller probability thus giving it less weight or influence. Several examples follow:

On game day, if a college football team wins, its bookstore estimates that it will sell 60 of the team's caps. However, if the team loses, they estimate that they will sell only 25 of the team's caps. If the team wins 60% of its games, find the bookstore's expected number of team caps sold.

	A	P
win	60	.60
lose	25	.40

E = 60(.60) + 25(.40)
= 36 + 10
= 46

A player spins the spinner shown at the right and wins the dollar amount shown. Find the player's expected winnings.



A	P
1	1/4
3	1/4
8	1/2

E = 1(1/4) + 3(1/4) + 8(1/2)
= 1/4 + 3/4 + 4
= 5 (or $5)

A multiple choice test contains four choices (a,b,c,d). Five points are awarded for a correct answer, but two points are deducted for an incorrect answer. Is it to a test taker's advantage to guess?

	A	P
guess right	5	1/4
guess wrong	-2	3/4

E = 5(1/4) + -2(3/4)
= 5/4 + -6/4
= -1/4

Since the expected value is negative, the test taker should not guess.

It costs an overnight delivery service $7 to deliver a package. So to make a profit, they charge customers $9 for delivery. However, if delivery is late, there is no charge to the customer. If only 2% of packages are delivered late, find the service's expected net gain per package.

	A	P	A	P
on time	9 - 7 = 2	98% = .98	2	.98
late	0 - 7 = -7	2% = .02	-7	.02

E = 2(.98) + -7(.02)
= 1.96 + -.14
= 1.82 (or $1.82)

In a game of chance, a player blindly selects (without replacement) a pair of bills from a box containing four bills ($1, $2, $5, $10) and gets to keep the larger of the two bills. If the game costs $8 to play, find the player's expected net winnings. **[first we must list the possible pairs]**

bills selected	net winnings
1,2	2 - 8 = -6
1,5	5 - 8 = -3
1,10	10 - 8 = 2
2,5	5 - 8 = -3
2,10	10 - 8 = 2
5,10	10 - 8 = 2

A	P
-6	1/6
-3	2/6
2	3/6

E = -6(1/6) + -3(2/6) + 2(3/6)
= -6/6 + -6/6 + 6/6
= -6/6
= -1 (or -$1)