Guided Notes

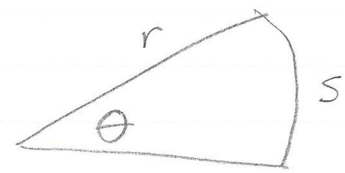
Section 10.4

Areas and Arc lengths in Polar Coordinates

Recall area of a sector of a circle $A = \frac{1}{2}r^2\theta$

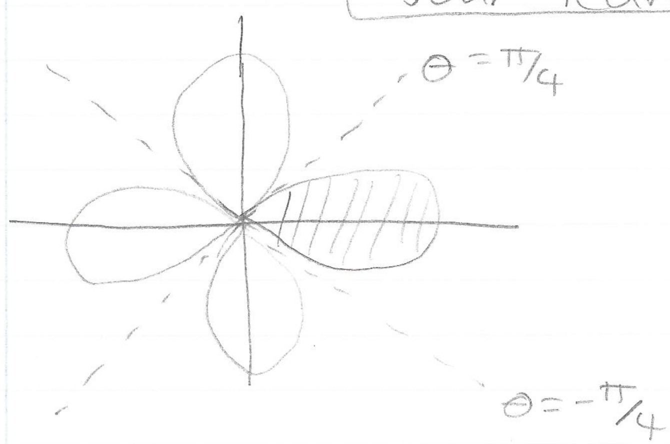
Radians only

$$\boxed{A = \int_a^b \frac{1}{2}r^2\,d\theta}$$



$\underline{s = r\theta}$

Example 1 Find the area of one loop of the four leaved rose $r = \cos 2\theta$.



note $\cos 2\theta = 0$

$$2\theta = \pi/2,\ 3\pi/2,\ 5\pi/2, \ldots$$
$$\theta = \frac{\pi}{4},\ \frac{3\pi}{4},\ \frac{5\pi}{4}, \ldots$$

the long part found when $\cos 2\theta = 1$

$$A = \int_{-\pi/4}^{\pi/4} \frac{1}{2}r^2\,d\theta = \int_{-\pi/4}^{\pi/4} \frac{1}{2}(\cos^2 2\theta)\,d\theta$$ use identity

$$= \frac{1}{2}\int_{-\pi/4}^{\pi/4} \frac{1}{2}(1+\cos 4\theta)\,d\theta = \frac{1}{2}\left[\theta + \frac{1}{4}\sin 4\theta\right|_0^{\pi/4}$$

$$= \frac{\pi}{8}$$

Arc Length

$$\boxed{L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta}$$

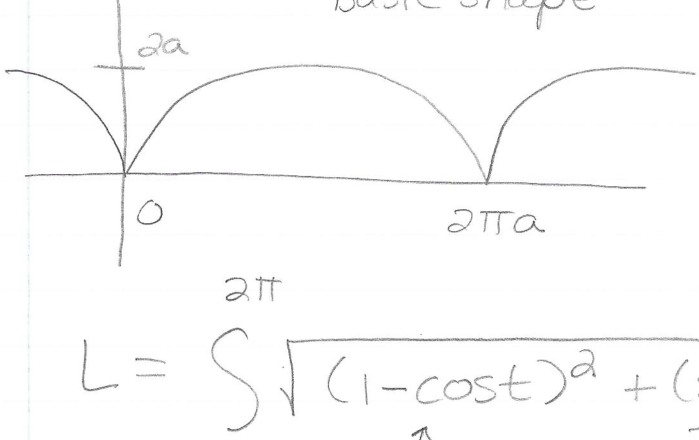
$$\boxed{L = \int_a^b \sqrt{(f(\theta))^2 + (f'(\theta))^2}\, d\theta}$$

Example

Recall from before

Find the length of one arc of the cycloid that has parametric equations $x = t - \sin t$ and $y = 1 - \cos t$

basic shape



$x = at - a\sin t$
$y = a - a\cos t$

$$L = \int_0^{2\pi} \sqrt{\underbrace{(1-\cos t)^2}_{\left(\frac{dx}{dt}\right)^2} + \underbrace{(\sin t)^2}_{\left(\frac{dy}{dt}\right)^2}}\, dt = \boxed{8}$$

(r, θ)

Example Find the length of the spiral having polar coordinates $r = e^{\theta/2}$ from $(\sqrt{e}, 1)$ to $(e, 2)$

$$L = \int_a^b \sqrt{(f(\theta))^2 + (f'(\theta))^2}\, d\theta$$

$$f(\theta) = r = e^{\theta/2}$$

$$f'(\theta) = r' = \tfrac{1}{2} e^{\theta/2}$$

$$L = \int_1^2 \sqrt{(e^{\theta/2})^2 + (\tfrac{1}{2} e^{\theta/2})^2}\, d\theta$$

$$L = \int_1^2 \sqrt{e^\theta + \tfrac{1}{4} e^\theta}\, d\theta$$

$$L = \int_r^2 \sqrt{\tfrac{5}{4} e^\theta}\, d\theta = \int_1^2 \frac{\sqrt{5}}{2} e^{\theta/2}\, d\theta$$

$$= \frac{\sqrt{5}}{2} \cdot 2\, e^{\theta/2} \Big|_1^2$$

$$= \sqrt{5}\left[e^1 - e^{1/2}\right] = \boxed{\sqrt{5}\,(e - \sqrt{e})}$$