Guided Notes Section 10.6	Conic Sections in Polar Coordinates

The Big Idea:

$\boxed{\text{Theorem}}$ Let F be a fixed point (called the <u>focus</u>) and l be a fixed line (called the <u>directrix</u>) in a plane. Let e be a fixed positive number (called the <u>eccentricity</u>). Then the set of all points P in the plane such that

$$\frac{|PF| \text{ (distance from F)}}{|Pl| \text{ (distance from } l)} = e$$

is a conic section.

specifically

a) an ellipse if $e < 1$

b) a parabola if $e = 1$

c) a hyperbola if $e > 1$

Converted format of equations for conic sections

$$\boxed{r = \frac{ed}{1 \pm e\cos\theta} \quad \text{or} \quad r = \frac{ed}{1 \pm e\sin\theta}}$$

$\cos\theta \rightarrow$ symmetry WRT to x

$\sin\theta \rightarrow$ symmetry WRT y

d = directrix

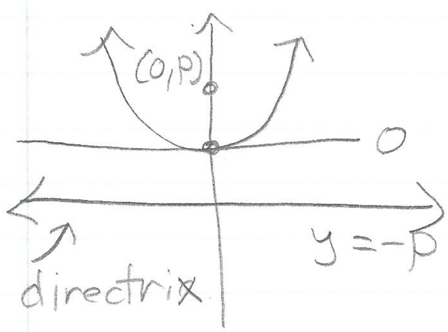
recall from parabolas

directrix $y = -p$ or $x = -p$

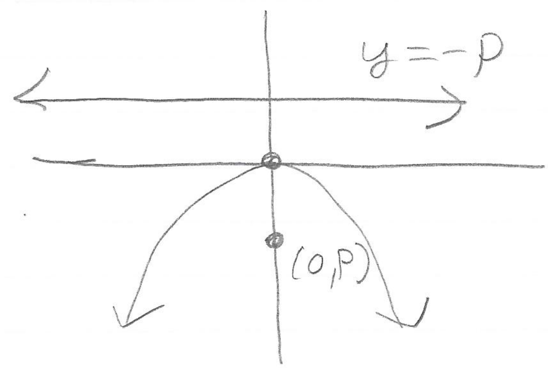
the directrix is the same distance (opposite direction) of the focus point

Visuals

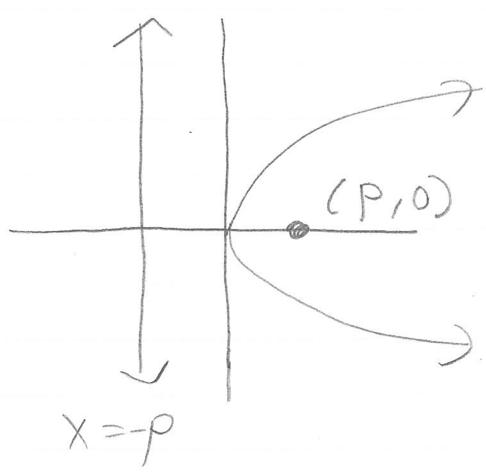
$\boxed{x^2 = 4py \ (p>0)}$



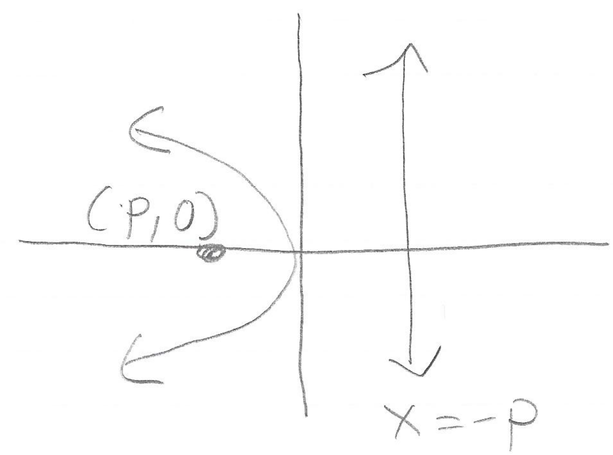
$\boxed{x^2 = 4py \ (p<0)}$



$\boxed{y^2 = 4px \ (p>0)}$



$\boxed{y^2 = 4px \ (p<0)}$



other useful information

$\boxed{e<1}$ ellipse

$$c^2 = a^2 - b^2$$

$$e = \frac{c}{a}$$

hyperbola

$\boxed{e>1}$

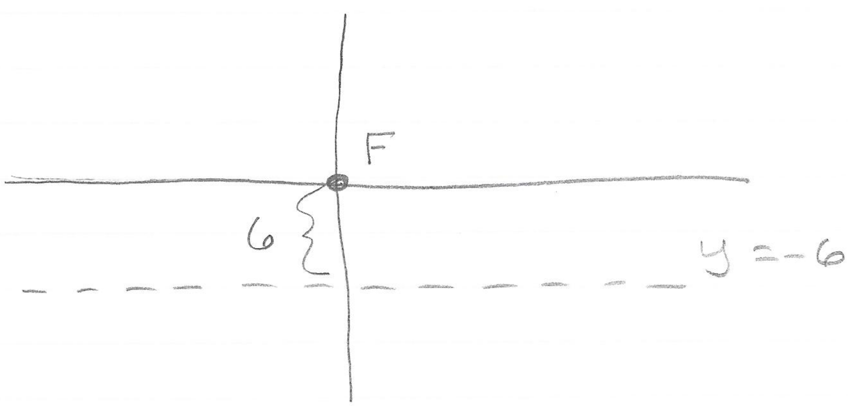
$$c^2 = a^2 + b^2$$

$$e = \frac{c}{a}$$

Example 1

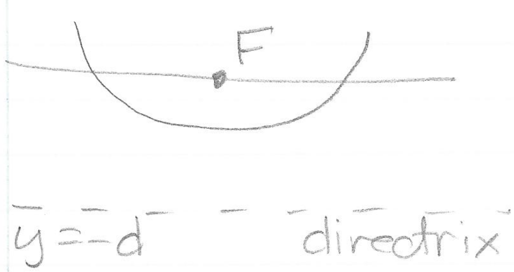
Find a polar equation for a parabola that has a focus at the origin and whose directrix is the line $y=-6$

parabola $\rightarrow$ $\boxed{e=1}$
focus $\rightarrow$ $(0,0)$
directrix $\rightarrow$ $y=-6$ $\rightarrow$ $\boxed{d=6}$



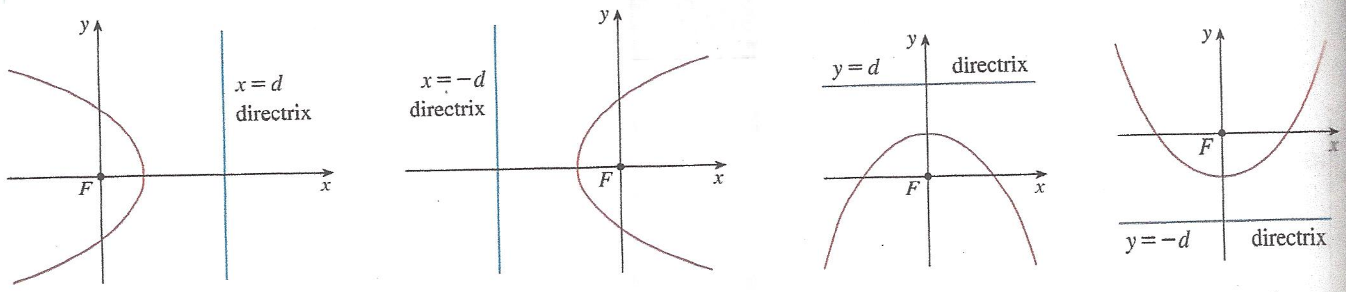
we choose (from our 4 options on page 684)

$$r=\frac{ed}{1-e\sin\theta}$$



$$r=\frac{(1)(+6)}{1-(1)\sin\theta}$$

$$\boxed{r=\frac{6}{1-\sin\theta}}$$



(a) $r = \dfrac{ed}{1 + e\cos\theta}$ (b) $r = \dfrac{ed}{1 - e\cos\theta}$ (c) $r = \dfrac{ed}{1 + e\sin\theta}$ (d) $r = \dfrac{ed}{1 - e\sin\theta}$

$\boxed{\text{Example 2}}$ Describe and sketch a graph of the equation

$$r = \frac{10}{3 - 2\cos\theta}$$

first we need to match one of the formats above

$$r = \frac{\frac{10}{3}}{1 - \frac{2}{3}\cos\theta}$$

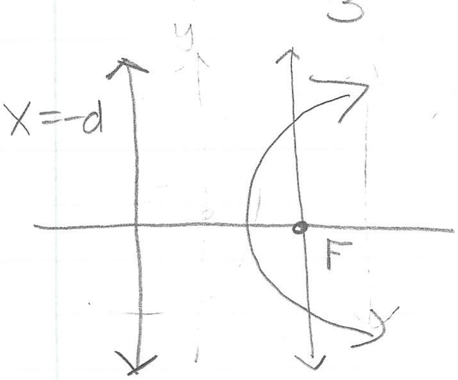
(form b)

$\boxed{e = \frac{2}{3}}$ $ed = \frac{10}{3}$

$d = \frac{10}{3} \cdot \frac{3}{2} = 5$

$\boxed{d = 5}$

since $\frac{2}{3} < 1 \Rightarrow \boxed{\text{ellipse}}$



with focus F and major axis along the polar axis.

major axis → X since $d = -x$

polar

To find the endpoints of the major axis, set $\theta = 0$ and $\theta = \pi$

$\boxed{\theta = 0}$

$$r = \frac{10}{3 - 2\cos(0)} = \frac{10}{3-2} = 10 \qquad (10, 0) = (r, \theta)$$

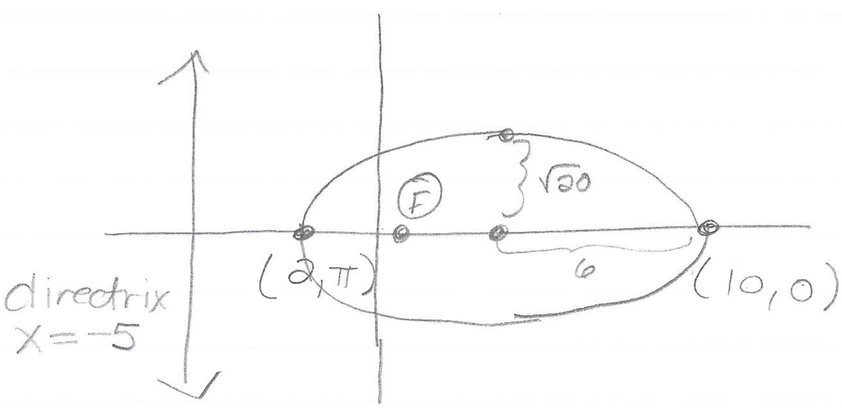
$\boxed{\theta = \pi}$

$$r = \frac{10}{3 - 2\underbrace{\cos(\pi)}_{-1}} = \frac{10}{3+2} = 2 \qquad (2, \pi) = (r, \theta)$$

The center = midpoint of $(2,0)$ & $(10,\pi)$

$\Rightarrow$ distance $= 6$ or $\boxed{a = 6}$

$\boxed{\text{basic sketch}}$



also $e = \frac{c}{a}$

focus

we know $e = \frac{|dF|}{|dl|}$

directrix

$$\frac{2}{3} = \frac{c}{a}$$

$$\frac{2}{3} = \frac{c}{6}$$

$\boxed{c = 4}$

so

$c^2 = a^2 - b^2$

$4^2 = 6^2 - b^2$

$16 = 36 - b^2$

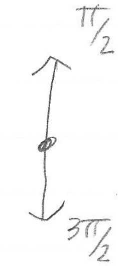
$-20 = -b^2$

$\boxed{\sqrt{20} = b}$

length of minor axis

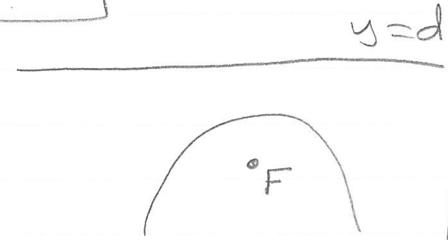
Example 3 Describe and sketch the graph of the equation

$$r = \frac{10}{2+3\sin\theta}$$



$$\boxed{r} = \frac{\frac{10}{2}}{\frac{2}{2}+\frac{3\sin\theta}{2}} = \boxed{\frac{5}{1+\frac{3}{2}\sin\theta}}$$

major axis $\Rightarrow y$



match $\quad \dfrac{ed}{1+e\sin\theta}$

endpoints $\frac{\pi}{2}, \frac{3\pi}{2}$

$ed = 5 \qquad e = \frac{3}{2} \qquad \left(\frac{3}{2}\right)d = 5$

$d = \frac{5(2)}{3} = \frac{10}{3}$

$$\boxed{d = \frac{10}{3}} \qquad \boxed{e = \frac{3}{2}}$$

$\varepsilon > 0$
hyperbola

what do we need to know?

center, a, b
asymptotes
endpoints

$\theta = \frac{\pi}{2} \quad r = \dfrac{5}{1+\frac{3}{2}\sin\left(\frac{\pi}{2}\right)} = 2$

$\theta = \frac{3\pi}{2} \quad r = \dfrac{5}{1+\frac{3}{2}\sin\left(\frac{3\pi}{2}\right)} = -10$

endpoints
$\left(2, \frac{\pi}{2}\right)$
$\left(-10, \frac{3\pi}{2}\right)$

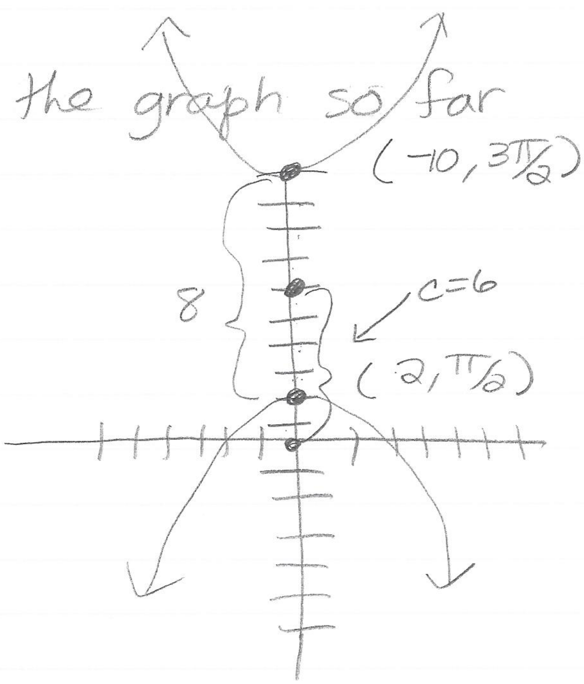
Summary of info, so far

$e = \frac{3}{2} > 0$ hyperbola with focus at the poles.

$\sin\theta \longrightarrow$ transverse axis of the hyperbola is $\perp$ to the polar axis.

vertices: $(2, \frac{\pi}{2})$ $(-10, \frac{3\pi}{2})$

the graph so far



$2a = 8$

$\boxed{a = 4}$

when $e > 1$

$e = \frac{c}{a}$

$c^2 = a^2 + b^2$

$\frac{3}{2} = \frac{c}{4}$

$\frac{12}{2} = \boxed{c = 6}$

$b^2 = c^2 - a^2$

$b^2 = 36 - 16$

$b = \pm\sqrt{20}$

asymptotes $y = \pm \frac{a}{b} x$

Example 4

Sketch the graph of the equation

$$r = \frac{15}{4 - 4\cos\theta}$$

$$r = \frac{\frac{15}{4}}{\frac{4}{4} - \frac{4}{4}\cos\theta} \Rightarrow \boxed{r = \frac{\frac{15}{4}}{1 - \cos\theta}}$$

$e = 1$
parabola

parabola with focus point at the pole

For parabolas, you can find a good rough sketch by plotting a few points

θ	0	$\pi/2$	π	$3\pi/2$
r	und	$15/4$ $(3\frac{3}{4})$	$15/8$ $(1\frac{7}{8})$	$15/4$ $(3\frac{3}{4})$