Section 7.3

The second translational theorem

If $F(s) = \mathcal{L}\{f(t)\}$ and $a > 0$, then

$$\mathcal{L}\{f(t-a)\mathcal{U}(t-a)\} = e^{-as}F(s)$$

with
$$\mathcal{U}(t-a) = \begin{cases} 0 & 0 \le t < a \\ 1 & t \ge a \end{cases}$$

unit step function

Example 1

$$\mathcal{L}^{-1}\left\{\frac{1}{s-4}e^{-2s}\right\} = \boxed{e^{4(t-2)}\mathcal{U}(t-2)}$$

$$\mathcal{L}\{f(t)\} = \frac{1}{s-4}$$

$$f(t) = e^{4t}$$

shift
$a = 2$
$\Rightarrow t-2$

Example 2

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}e^{-\frac{\pi s}{2}}\right\} = \boxed{\cos 3\left(t-\frac{\pi}{2}\right)\mathcal{U}\left(t-\frac{\pi}{2}\right)}$$

$a=3$

$$f(t) = \cos 3t$$

$a = \frac{\pi}{2}$

$\left(t - \frac{\pi}{2}\right)$

Example 3 #44 from the book

$$\boxed{\mathcal{L}^{-1}\left\{\frac{(1+e^{-2s})^2}{s+2}\right\}}$$

$$\mathcal{L}^{-1}\left\{\frac{1+2e^{-2s}+e^{-4s}}{s+2}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s+2}+\frac{2}{s+2}e^{-2s}+\frac{1}{s+2}e^{-4s}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}+2\mathcal{L}^{-1}\left\{\frac{1}{s+2}e^{-2s}\right\}+\mathcal{L}^{-1}\left\{\frac{1}{s+2}e^{-4s}\right\}$$

e^{-2t} + e^{-2t} $a=2$ $(t-2)$ e^{-2t} $a=4$ $t-4$

$$=\boxed{e^{-2t}+e^{-2(t-2)}\mathcal{U}(t-2)+e^{-2(t-4)}\mathcal{U}(t-4)}$$

Example 4 (# 68)

$$y'' - 5y' + 6y = \mathcal{U}(t-1) \qquad y(0)=0, \quad y'(0)=1$$

$$\mathcal{L}\{y''\} - 5\mathcal{L}\{y'\} + 6\mathcal{L}\{y\} = \mathcal{L}\{\mathcal{U}(t-1)\}$$

$$s^2\mathcal{L}\{y\} - s\,y(0) - y'(0) - 5[s\mathcal{L}\{y\} - y(0)]$$
$$+ 6\mathcal{L}\{y\} = \mathcal{L}\{\mathcal{U}(t-1)\}$$

$$s^2\mathcal{L}\{y\} - s\cdot \cancel{y(0)}^{0} - \cancel{y'(0)}^{1} - 5[s\mathcal{L}\{y\} - \cancel{y(0)}^{0}] + 6\mathcal{L}\{y\}$$
$$= \mathcal{L}\{\underbrace{1}_{\mathcal{L}\{1\}=\frac{1}{s}} \cdot \underbrace{\mathcal{U}(t-1)}_{a=1}\}$$

$$s^2\mathcal{L}\{y\} - 1 - 5s\mathcal{L}\{y\} + 6\mathcal{L}\{y\} = \frac{e^{-s}}{s}$$

$$(s^2 - 5s + 6)\mathcal{L}\{y\} = \frac{e^{-s}}{s} + 1$$

$$\mathcal{L}\{y\} = \left(\frac{e^{-s}}{s} + 1\right)\left(\frac{1}{s^2-5s+6}\right)$$

$$\mathcal{L}\{y\} = e^{-s}\left(\frac{1}{s(s-2)(s-3)}\right) + \frac{1}{(s-2)(s-3)}$$

$$y = \underbrace{\mathcal{L}^{-1}\left\{\frac{1}{s(s-2)(s-3)}e^{-s}\right\}}_{\text{partial fractions (part 1)}} + \underbrace{\mathcal{L}^{-1}\left\{\frac{1}{(s-2)(s-3)}\right\}}_{\text{partial fractions. (part 2)}}$$

$\boxed{\text{part 1}}$

$$\frac{1}{s(s-2)(s-3)} = \frac{A}{s} + \frac{B}{s-2} + \frac{C}{s-3}$$

$$A(s-2)(s-3) + B(s)(s-3) + C(s)(s-2) = 1$$
$$A(s^2-5s+6) + B(s^2-3s) + C(s^2-2s) = 1$$

$$s^2(A+B+C) + s(-5A-3B-2C) + (6A) = 1$$

$$6A = 1 \qquad \boxed{A = 1/6}$$

$$A+B+C=0$$
$$-5A-3B-2C=0$$

$$\frac{1}{6} + B + C = 0$$
$$\boxed{B+C = -\frac{1}{6}}$$

$$-5\left(\frac{1}{6}\right) - 3B - 2C = 0$$
$$\boxed{-3B - 2C = \frac{5}{6}}$$

$$B = -\frac{1}{6} - C$$

$$-3\left(-\frac{1}{6} - C\right) - 2C = \frac{5}{6}$$

$$\frac{3}{6} + 3C - 2C = \frac{5}{6}$$

$$\boxed{C = \frac{2}{6} = \frac{1}{3}}$$

$$B + C = -\frac{1}{6}$$
$$B + \frac{1}{3} = -\frac{1}{6}$$
$$B = -\frac{1}{6} - \frac{2}{6} = -\frac{3}{6} = -\frac{1}{2}$$

$$\boxed{B = -\frac{1}{2}}$$

so $\mathcal{L}^{-1}\left\{\frac{1}{s(s-2)(s-3)} e^{-s}\right\}$

$$= \mathcal{L}^{-1}\left\{\left(\frac{1}{6}\cdot\frac{1}{s} - \frac{1}{2}\frac{1}{s-2} + \frac{1}{3}\frac{1}{s-3}\right)e^{-s}\right\}$$

$$= \boxed{\left(\frac{1}{6} - \frac{1}{2}e^{2(t-1)} + \frac{1}{3}e^{3(t-1)}\right)\mathcal{U}(t-1)}$$

$\boxed{\text{part 2}}$ $\dfrac{1}{(s-2)(s-3)} = \dfrac{A}{s-2} + \dfrac{B}{s-3}$

$A(s-3) + B(s-2) = 1$

$As - 3A + Bs - 2B = 1$

$s(A+B) + (-3A - 2B) = 1$

$A + B = 0 \qquad -3A - 2B = 1$

$A = -B \qquad -3(-B) - 2B = 1$

$3B - 2B = 1$

$\boxed{A = -1} \qquad \boxed{B = 1}$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-2)(s-3)}\right\} = \mathcal{L}^{-1}\left\{\frac{-1}{s-2} + \frac{1}{s-3}\right\}$$

$$= \boxed{-e^{2t} + e^{3t}}$$

So, all together,

$$\boxed{y = \left(\frac{1}{6} - \frac{1}{2}e^{2(t-1)} + \frac{1}{3}e^{3(t-1)}\right)u(t-1) - e^{2t} + e^{3t}}$$

note — alternate notation

$$u(t-a) = u_a(t)$$

(Heaviside function)

practice exercises

(1) $\mathcal{L}^{-1}\left\{\dfrac{e^{-3s}}{s^2}\right\}$

(2) $\mathcal{L}^{-1}\left\{\dfrac{e^{-s}}{s+2}\right\}$

(3) $\mathcal{L}^{-1}\left\{\dfrac{e^{-\pi s}}{s^2+1}\right\}$

(4) $\mathcal{L}^{-1}\left\{\dfrac{1-e^{-2\pi s}}{s^2+1}\right\}$

(5) $\mathcal{L}^{-1}\left\{\dfrac{s(1+e^{-3s})}{s^2+\pi^2}\right\}$

answers

(1) $f(t) = (t-3)\,u(t-3)$

(2) $f(t) = e^{-2(t-1)}\,u(t-1)$

(3) $f(t) = \sin(t-\pi)\,u(t-\pi)$

(4) $f(t) = \left[1 - u(t-2\pi)\right]\sin(t)$

(5) $f(t) = \left[1 - u(t-3)\right]\cos(\pi t)$

solve

(6) $\mathcal{L}\{-28\,u(t-3)\}$

(7) $\mathcal{L}\{3u(t-8) - u(t-4)\}$

(8) $\mathcal{L}\{-42e^{t-4}\,u(t-4)\}$

answers

(6) $-28e^{-3s}s^{-1}$

(7) $\left(3e^{-8s} - e^{-4s}\right)\left(\frac{1}{s}\right)$

(8) $-\left(\dfrac{42e^{-4s}}{s-1}\right)$

$\boxed{\text{Solve each initial value problem}}$

(9) $\dfrac{dy}{dt} + 9y = u(t-5) \qquad y(0) = -2$

(10) $\dfrac{dy}{dt} = -y + u(t-2)e^{-2(t-2)} \qquad y(0) = 1$

(11) $\dfrac{dy}{dt} = -y + u_1(t)(t-1) \qquad y(0) = 2$

$\boxed{\text{answers}}$

(9) $y = -2e^{-9t} + \frac{1}{9}u_5(t) - \frac{1}{9}u_5(t)\,e^{-9(t-5)}$

(10) $y = e^{-t} + u_2(t)\left(e^{-(t-2)} - e^{-2(t-2)}\right)$

(11) $y(t) = u_1(t)\left((t-2) + e^{-(t-1)}\right) + 2e^{-t}$