$\boxed{\text{Section 7.4}}$ Operational Properties II

$\boxed{\text{Formulas to know}}$

$\boxed{\text{Derivatives of Transforms}}$

If $F(s) = \mathcal{L}\{f(t)\}$ and $n = 1, 2, 3, \ldots$ then

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$$

$\boxed{\text{Example}}$ (#2 from book)

$\mathcal{L}\{t^3 e^t\}$ $\quad F(s) = \mathcal{L}\{e^t\} = \dfrac{1}{s-1} = (s-1)^{-1}$

third derivative of transform

$$\frac{d}{ds}(F(s)) = -1(s-1)^{-2}$$

$$\frac{d^2}{ds^2}(F(s)) = 2(s-1)^{-3}$$

$$\frac{d^3}{ds^3}(F(s)) = -6(s-1)^{-4}$$

$$\mathcal{L}\{t^3 e^t\} = (-1)^3 \frac{d^3}{ds^3} F(s)$$

$$= (-1)\left(\frac{-6}{(s-1)^4}\right) = \boxed{\frac{6}{(s-1)^4}}$$

* watch out for negative signs. *

practice problems

1. $\mathcal{L}\{t \sin 3t\}$
2. $\mathcal{L}\{te^{2t}\cos 3t\}$
3. $\mathcal{L}\{te^{-t}\}$
4. $\mathcal{L}\{t^2 e^{2t}\}$
5. $\mathcal{L}\{t^2 \cos 2t\}$
6. $\mathcal{L}\{te^{-t}\sin^2 t\}$

answers

1. $\dfrac{6s}{(s^2+9)^2}, \quad s>0$
2. $\dfrac{(s^2-4s-5)}{(s^2-4s+13)^2}, \quad s>0$
3. $\dfrac{1}{(s+1)^2}$
4. $\dfrac{2}{(s-2)^3}$
5. $\dfrac{(2s^3-24s)}{(s^2+4)^3}, \quad s>0$
6. $\dfrac{2(3s^2+6s+7)}{(s+1)^2(s^2+2s+5)^2}, \quad s>0$

The convolution theorem

$$f*g = \int_0^t f(\tau)g(t-\tau)\,d\tau$$

$$\mathcal{L}\{f*g\} = \mathcal{L}\{f(t)\}\,\mathcal{L}\{g(t)\} = F(s)\,G(s)$$

when $f(t)$ and $g(t)$ are piecewise continuous functions

$\boxed{\text{Example}}$ (#20 book)

$$\boxed{\mathcal{L}\{t^2 * te^t\}}$$

$$= \mathcal{L}\{t^2\} \cdot \mathcal{L}\{te^t\}$$

$$= \frac{2}{s^3} \cdot \frac{1}{(s-1)^2} = \boxed{\frac{2}{s^3(s-1)^2}}$$

$\boxed{\text{Example}}$ (#22 book)

$$\boxed{\mathcal{L}\{e^{2t} * \sin t\}}$$

$$\mathcal{L}\{e^{2t}\} \cdot \mathcal{L}\{\sin t\}$$

$$= \frac{1}{s-2} \cdot \frac{1}{s^2+1} = \boxed{\frac{1}{(s-2)(s^2+1)}}$$

$\boxed{\text{Example}}$ (#24 book)

$$\boxed{\mathcal{L}\left\{ \int_0^t \cos\tau \, d\tau \right\}}$$

recall

$$f*g = \int_0^t f(\tau)g(t-\tau)d\tau$$

let $g(t-\tau) = 1 \qquad f(\tau) = \cos\tau$

so $\mathcal{L}\left\{ \int_0^t \cos\tau \, d\tau \right\} = \mathcal{L}\{\cos t * 1\}$

$$= \mathcal{L}\{\cos t\} \cdot \mathcal{L}\{1\}$$

$$= \frac{s}{s^2+1} \cdot \frac{1}{s}$$

$$= \boxed{\frac{1}{s^2+1}}$$

$\boxed{\text{Example}}$ (#26 from book)

$$\boxed{\mathcal{L}\left\{\int_0^t \tau \sin\tau \, d\tau\right\}}$$

notice both parts involve τ

$$\mathcal{L}\left\{\int_0^t f(\tau)\, g(t-\tau)\, d\tau\right\} = \mathcal{L}\{1 * (t\sin t)\}$$

$f(\tau) \to \tau \sin\tau$ uses the derivative of a transform

$g(t-\tau) \to 1$

$$= \mathcal{L}\{1\} \cdot \mathcal{L}\{t\sin t\} = \frac{1}{s} \cdot \left(-\frac{d}{ds} \frac{1}{s^2+1}\right)$$

$$= \frac{-1}{s} \cdot \frac{-2s}{(s^2+1)^2} = \boxed{\frac{2s}{s(s^2+1)^2}}$$

$\boxed{\text{Example}}$ (#30 from book)

$$\boxed{\mathcal{L}\left\{ t \int_0^t \tau e^{-\tau}\, d\tau \right\}}$$

derivative of a transform !

$$= -\frac{d}{ds}\, \underbrace{\mathcal{L}\left\{ \int_0^t \tau e^{-\tau}\, d\tau \right\}}_{\text{another derivative of a transform similar to \#26}}$$

$$= -\frac{d}{ds}\left[\frac{1}{s} \cdot \frac{1}{(s+1)^2} \right] = \boxed{\frac{3s+1}{s^2(s+1)^3}}$$

$\boxed{\text{Example}}$ (# 32 from book)

$$\boxed{\mathcal{L}^{-1}\left\{ \frac{1}{s^2(s-1)} \right\}} = \mathcal{L}^{-1}\left\{ \underbrace{\frac{1}{s^2}}_{\nearrow t} \cdot \underbrace{\frac{1}{s-1}}_{\nearrow e^t} \right\}!$$

Convolution !

recall
$\mathcal{L}\{f*g\} = \mathcal{L}\{f\} \cdot \mathcal{L}\{g\}$

so $f*g = \mathcal{L}^{-1}\{ \mathcal{L}\{f\} \cdot \mathcal{L}\{g\} \}$

$$= t * e^t$$

$$= \int_0^t \tau \cdot e^{t-\tau}\, d\tau = \underline{\text{Integration by parts}}$$

$$\int_0^t \tau e^{t-\tau}\,d\tau$$

	D	I
$+$	τ	$e^{t-\tau}$
$-$	1	$-e^{t-\tau}$
$+$	0	$e^{t-\tau}$

$$= -\tau e^{t-\tau} - e^{t-\tau}\Big|_0^t$$

$$= \left(-te^{t-t} - e^{t-t}\right) - \left(0 - e^{t-0}\right)$$

$$= \left(-te^{0} - e^{0}\right) - \left(-e^{t}\right)$$

$$= \boxed{-t - 1 + e^{t}}$$

Alternate approach for #32

using $\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \dfrac{F(s)}{s} = \dfrac{1}{s}F(s)$

$$\mathcal{L}\left\{\frac{1}{s^2(s-1)}\right\} = \mathcal{L}\left\{\frac{\left(\frac{1}{s(s-1)}\right)}{s}\right\} = \mathcal{L}\left\{\frac{1}{s}\cdot\frac{1}{s(s-1)}\right\}$$

so $\mathcal{L}\{f(t)\} = \dfrac{1}{s(s-1)}$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s(s-1)}\right\} = \mathcal{L}^{-1}\left\{\frac{A}{s} + \frac{B}{s-1}\right\}$$

practice problems

Solve each equation

1. $y' - 5y = e^{-2t}$ $y(0) = 1$
2. $y' - 4y = \cos 2t$ $y(0) = -2$
3. $y'' + 2y' + 2y = \cos 2t$ $y(0) = 1$, $y'(0) = 0$
4. $y'' + 3y' + 5y = e^{-t} + t$ $y(0) = -1$, $y'(0) = 0$
5. $y' - y = t^2 e^{-2t}$ $y(0) = 0$
6. $y'' + y' + 2y = e^{-t}\cos 2t$, $y(0) = 1$, $y'(0) = -1$

answers

1. $\dfrac{(s+3)}{(s-5)(s+2)}$
2. $\dfrac{(-2s^2 + s - 8)}{(s-4)(s^2+4)}$
3. $\dfrac{s^3 + 2s^2 + 5s + 8}{(s^2+4)(s^2+2s+2)}$
4. $\dfrac{(6s^2 + 8s + 5)}{s^2(s+1)(s^2+3s+5)}$
5. $\dfrac{1}{(s-1)(s+2)^2}$
6. $\dfrac{(-2s^2 + 9s - 12)}{(s-2)(s^2-4s+5)}$

practice problems

① $\mathcal{L}^{-1}\left\{\dfrac{1}{s^3(s+3)}\right\}$

② $\mathcal{L}^{-1}\left\{\dfrac{1}{(s^2+4)(s+1)}\right\}$

③ $\mathcal{L}^{-1}\left\{\dfrac{e^{-\pi s}}{s^3+s}\right\}$

answers

① $\frac{1}{54}\left(2-2e^{-3t}-6t+9t^2\right)$

③ $(1+\cos t)\,\mathcal{U}(t-\pi)$

② $\frac{1}{10}\left(2e^{-t}-2\cos 2t+\sin 2t\right)$