Section 7.2

Inverse LaPlace Transforms and Transforms of Derivatives

Inverse Formulas

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1 \qquad \mathcal{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\} = \sin kt$$

$$\mathcal{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n \qquad \mathcal{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\} = \cos kt$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at} \qquad \mathcal{L}^{-1}\left\{\frac{k}{s^2-k^2}\right\} = \sinh kt$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t \qquad \mathcal{L}^{-1}\left\{\frac{s}{s^2-k^2}\right\} = \cosh kt$$

$$\mathcal{L}^{-1}\left\{\frac{n!}{(s-a)^{n+1}}\right\} = t^n e^{at}$$

Examples

① $\boxed{\mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\}}$

formula: $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$

so $n+1=4$
$\boxed{n=3}$ we need $3! = 6$ in the numerator

so $\mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\} = \frac{1}{6}\mathcal{L}^{-1}\left\{\frac{6}{s^4}\right\} = \boxed{\frac{1}{6}t^3}$

② $\boxed{\mathcal{L}^{-1}\left\{\frac{(s+2)^2}{s^3}\right\}} = \mathcal{L}^{-1}\left\{\frac{s^2+4s+4}{s^3}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}+\frac{4}{s^2}+\frac{4}{s^3}\right\}$

$= \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{4}{s^2}\right\} + \mathcal{L}^{-1}\left\{\frac{4}{s^3}\right\}$

$n+1=1$, $n=0$, need $(0!)$ $\quad$ $n+1=2$, $n=1$, need $(1!)$ $\quad$ $n+1=3$, $n=2$, need $(2!)$

(2) continued

$$= \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + 4\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + 2\mathcal{L}^{-1}\left\{\frac{2}{s^3}\right\}$$

$$= \boxed{1 + 4t + 2t^2}$$

(3) $\boxed{\mathcal{L}^{-1}\left\{\frac{s+1}{s^2-4s}\right\}} = \mathcal{L}^{-1}\left\{\frac{s+1}{s(s-4)}\right\}$ need partial fractions

$$= \mathcal{L}^{-1}\left\{\frac{A}{s} + \frac{B}{s-4}\right\}$$

$\boxed{a=4}$

fundamental transform

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$$

$$= \boxed{A(1) + Be^{4t}}$$

find A & B

$$A(s-4) + B(s) = s+1$$
$$As - 4A + Bs = s+1$$
$$s(A+B) - 4A = s+1$$

$$A+B = 1$$
$$-\frac{1}{4} + B = 1$$
$$\boxed{B = \frac{5}{4}}$$

$$-4A = 1$$
$$\boxed{A = -\frac{1}{4}}$$

so $\mathcal{L}^{-1}\left\{\frac{s+1}{s^2-4s}\right\} = \boxed{-\frac{1}{4} + \frac{5}{4}e^{4t}}$

(4) $\boxed{\mathcal{L}^{-1}\left\{\frac{6s+3}{s^4+5s^2+4}\right\}} = \mathcal{L}^{-1}\left\{\frac{6s+3}{(s^2+4)(s^2+1)}\right\}$

need to use partial fractions

the set-up

$$\mathcal{L}^{-1}\left\{\frac{As+B}{s^2+4}\right\} + \mathcal{L}^{-1}\left\{\frac{Cs+D}{s^2+1}\right\}$$

(4) $\boxed{\text{continued}}$

$$= \mathcal{L}^{-1}\left\{\frac{As}{s^2+4}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2B}{s^2+4}\right\} + \mathcal{L}^{-1}\left\{\frac{Cs}{s^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1D}{s^2+1}\right\}$$

ok ✓

part 1: $K=\sqrt{4}=2$

part 2: $K=\sqrt{4}=2$ (missing a 2)

part 3: $K=\sqrt{1}=1$

part 4: $K=\sqrt{1}=1$

$\boxed{\text{for parts 1 \& 3}}$ $\rightarrow$ $\mathcal{L}\{\cos Kt\} = \dfrac{s}{s^2+K^2}$

$\boxed{\text{for parts 2 \& 4}}$ $\rightarrow$ $\mathcal{L}\{\sin Kt\} = \dfrac{K}{s^2+K^2}$

$$= \boxed{A\cos 2t + \frac{B}{2}\sin 2t + C\cdot\cos t + D\sin t}$$

Find A, B, C, D

recall $\quad \dfrac{(As+B)}{s^2+4} + \dfrac{(Cs+D)}{s^2+1} = \dfrac{6s+3}{(s^2+4)(s^2+1)}$

$(As+B)(s^2+1) + (Cs+D)(s^2+4) = 6s+3$

$As^3 + As + Bs^2 + B + Cs^3 + 4Cs + Ds^2 + 4D = 6s+3$

$s^3(A+C) + s^2(B+D) + s(A+4C) + B+4D = 6s+3$

$A+C=0 \quad B+D=0 \quad A+4C=6 \quad B+4D=3$

$A=-C \quad B=-D$

$$= \boxed{-2\cos 2t - \frac{1}{2}\sin 2t + 2\cos t + \sin t}$$

$-C+4C=6$

$3C=6$

$\boxed{C=2}$

$\boxed{A=-2}$

$-D+4D=3$

$3D=3$

$\boxed{D=1}$

$\boxed{B=-1}$

Some practice problems for Inverse LaPlace Transforms

① $\mathcal{L}^{-1}\left\{\frac{1}{s^8}\right\}$

② $\mathcal{L}^{-1}\left\{\frac{1}{s+5}\right\}$

③ $\mathcal{L}^{-1}\left\{\frac{1}{(s+6)^3}\right\}$

④ $\mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\}$

⑤ $\mathcal{L}^{-1}\left\{\frac{1}{s^2+15s+56}\right\}$

⑥ $\mathcal{L}^{-1}\left\{\frac{s}{s^2-5s-14}\right\}$

⑦ $\mathcal{L}^{-1}\left\{\frac{s-2}{s^2-4s}\right\}$

⑧ $\mathcal{L}^{-1}\left\{\frac{1}{s^2-4s-12}\right\}$

⑨ $\mathcal{L}^{-1}\left\{\frac{2s^2-s+2}{s(s^2+1)}\right\}$

⑩ $\mathcal{L}^{-1}\left\{\frac{3s-4}{s(s-2)}\right\}$

⑪ $\mathcal{L}^{-1}\left\{\frac{s^3+s^2+4}{s^2(s^2+4)}\right\}$

⑫ $\mathcal{L}^{-1}\left\{\frac{s^3-s^2+2s-6}{s^5}\right\}$

Answers

① $\frac{1}{5040}t^7$

② e^{-5t}

③ $\frac{1}{2}t^2e^{-6t}$

④ $\frac{1}{3}\sin 3t$

⑤ $8e^{-8t}-7e^{-7t}$

⑥ $\frac{2}{9}e^{-2t}+\frac{1}{8}e^{6t}$

⑦ $\frac{1}{2}+\frac{1}{2}e^{4t}$

⑧ $\frac{-1}{8}e^{-2t}+\frac{1}{8}e^{6t}$

⑨ $2-\sin t$

⑩ $2+e^{2t}$

⑪ $t+\cos 2t$

⑫ $t-\frac{1}{2}t^2+\frac{1}{3}t^3-\frac{1}{4}t^4$