Section 7.2

Inverse transforms and transforms of derivatives

Formulas

Transforms of Derivatives

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0) = s\mathcal{L}\{f\} - f(0)$$
$$\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$$
$$\mathcal{L}\{f'''(t)\} = s^3F(s) - s^2f(0) - sf'(0) - f''(0)$$

$$\mathcal{L}\{f^{(n)}(t)\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots f^{n-1}(0)$$

Examples

Use LaPlace Transforms to Solve an initial value differential equation

Example 1

$$2\frac{dy}{dt} + y = 0 \quad \text{with} \quad y(0) = -3$$

$$2\mathcal{L}\left\{\frac{dy}{dt}\right\} + \mathcal{L}\{y\} = \mathcal{L}\{0\}$$

$$2\left[s\mathcal{L}\{y\} - \underbrace{y(0)}_{-3}\right] + \mathcal{L}\{y\} = 0$$

$$2s\mathcal{L}\{y\} + 6 + \mathcal{L}\{y\} = 0$$

$$2s\mathcal{L}\{y\} + \mathcal{L}\{y\} = -6$$

$$\mathcal{L}\{y\}(2s+1) = -6$$

$$\mathcal{L}\{y\} = \frac{-6}{2s+1}$$

$$y = \mathcal{L}^{-1}\left\{\frac{-6}{2s+1}\right\}$$

divide by 2

$$y = \mathcal{L}^{-1}\left\{\frac{-3}{s+\frac{1}{2}}\right\}$$

$$\boxed{y = -3e^{-\frac{1}{2}t}}$$

$\boxed{\text{Example 2}}$ $\quad y'' - 4y' = 6e^{3t} - 3e^{-t} \qquad y(0) = 1, \quad y'(0) = -1$

$$\mathcal{L}\{y''\} - 4\mathcal{L}\{y'\} = \mathcal{L}\{6e^{3t}\} - \mathcal{L}\{3e^{-t}\}$$

$$\left[s^2\mathcal{L}\{y\} - s\cancelto{1}{y(0)} - \cancelto{-1}{y'(0)}\right] - 4\left[s\mathcal{L}\{y\} - \cancelto{1}{y(0)}\right]$$
$$= \mathcal{L}\{6e^{3t}\} - \mathcal{L}\{3e^{-t}\}$$

$$s^2\mathcal{L}\{y\} - s + 1 - 4s\mathcal{L}\{y\} + 4 = \frac{6}{s-3} - \frac{3}{s+1}$$

(Isolate $\mathcal{L}\{y\}$)

$$\mathcal{L}\{y\}\left[s^2 - 4s\right] = \frac{6}{s-3} - \frac{3}{s+1} + s - 5.$$

$$\mathcal{L}\{y\} = \left[\frac{6}{s-3} - \frac{3}{s+1} + (s-5)\right]\frac{1}{s(s-4)}$$

$$\mathcal{L}\{y\} = \left[\frac{6}{s(s-3)(s-4)} - \frac{3}{s(s+1)(s-4)} + \frac{s-5}{s(s-4)}\right]$$

$$\boxed{y = \mathcal{L}^{-1}\left\{\underbrace{\frac{6}{s(s-3)(s-4)}}_{\text{part 1}} + \underbrace{\frac{-3}{s(s+1)(s-4)}}_{\text{part 2}} + \underbrace{\frac{s-5}{s(s-4)}}_{\text{part 3}}\right\}}$$

now, we have three sets of partial fractions to deal with! (combine them first!)

$$y = \mathcal{L}^{-1}\left\{\frac{6(s+1) - 3(s-3) + (s-5)(s-3)(s+1)}{s(s-3)(s-4)(s+1)}\right\}$$

$$y = \mathcal{L}^{-1}\left\{ \frac{6s+6-3s+9+(s-5)(s^2-2s-3)}{s(s-3)(s-4)(s+1)} \right\}$$

$$y = \mathcal{L}^{-1}\left\{ \frac{6s+6-3s+9+s^3-2s^2-3s-5s^2+10s+15}{s(s-3)(s-4)(s+1)} \right\}$$

$$y = \mathcal{L}^{-1}\left\{ \frac{+s^3-7s^2+10s+30}{s(s-3)(s-4)(s+1)} \right\}$$

$$y = \mathcal{L}^{-1}\left\{ \frac{A}{s} + \frac{B}{s-3} + \frac{C}{s-4} + \frac{D}{s+1} \right\}$$

$$\boxed{y = A(1) + Be^{3t} + Ce^{4t} + De^{-t}}$$

$$\boxed{\begin{array}{ll} A = 5/2 & \\ B = -2 & D = -3/5 \\ C = 11/10 & \end{array}}$$

$$A(s-3)(s-4)(s+1) + B(s)(s-4)(s+1) + C(s)(s-3)(s+1) + D(s)(s-3)(s-4) = s^3 - 7s^2 + 10s + 30$$

$A(s-3)(s^2-3s-4)$

$A(s^3-3s^2-4s$
$\quad -3s^2+9s+12)$

$A(s^3-6s^2+5s+12)$

$\boxed{As^3-6As^2+5As+12A}$

$B(s)(s^2-3s-4)$

$B(s^3-3s^2-4s)$

$\boxed{Bs^3-3Bs^2-4B\cdot s}$

$C(s)(s^2-2s-3)$

$C(s^3-2s^2-3s)$

$\boxed{Cs^3-2C\cdot s^2-3C\cdot s}$

$D(s)(s-3)(s-4)$

$D(s)(s^2-7s+12)$

$D(s^3-7s^2+12s)$

$\boxed{Ds^3-7Ds^2+12Ds}$

$s^3(A+B+C+D) = s^3$

$s^2(-6A-3B-2C-7D) = -7s^2$

$s(5A-4B-3C+12D) = 10s$

$(12A) = 30$

$12A = 30$

$\boxed{A} = \frac{30}{12} = \frac{10}{4} = \boxed{\frac{5}{2}}$

$\boxed{\text{Options for } 4\times4}$

1. $A + B + C + D = 1$
2. $-6A - 3B - 2C - 7D = -7$
3. $5A - 4B - 3C + 12D = 10$
4. $12A + 0B + 0C + 0D = 30$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ -6 & -3 & -2 & -7 & -7 \\ 5 & -4 & -3 & 12 & 10 \\ 12 & 0 & 0 & 0 & 30 \end{array}\right] \longrightarrow \begin{array}{c} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 5/2 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 11/10 \\ 0 & 0 & 0 & 1 & -3/5 \end{array}\right] \end{array} \begin{array}{c} A \\ B \\ C \\ D \end{array}$$

using calculator

By Hand (Expansion by minors) $\rightarrow$ Cramer's Rule

7 (1) $7A + 7B + 7C + \cancel{7D} = 7$
(2) $-6A - 3B - 2C - \cancel{7D} = -7$
$\boxed{A + 4B + 5C = 0}$

-12 (1) $-12A + -12B - 12C - \cancel{12D} = -12$
(3) $5A - 4B - 3C + \cancel{12D} = 10$
$\boxed{-7A - 16B - 15C = -2}$

$12A = 30$
$\boxed{A} = 30/12 = \boxed{5/2}$

$\frac{5}{2} + 4B + 5C = 0$

$5 + 8B + 10C = 0$

$\boxed{8B + 10C = -5}$

$-7\left(\frac{5}{2}\right) - 16B - 15C = -2$

$-35 - 32B - 30C = -4$

$\boxed{-32B - 30C = +35 - 4}$

$-32B - 30C = 31$

$3(8B + 10C = -5)$

$24B + \cancel{30C} = -15$
$-32B - \cancel{30C} = 31$
$-8B = 16$

$-8B = 16$
$\boxed{B = -2}$

$8B + 10C = -5$
$8(-2) + 10C = -5$
$-16 + 10C = -5$
$10C = 11$
$\boxed{C = 11/10}$

$A + B + C + D = 1$
$\frac{5}{2} - 2 + \frac{11}{10} + D = 1$
$\boxed{D = -\frac{3}{5}}$

$\boxed{\text{transforms of derivatives}}$ → continued

$\boxed{\text{Example 2}}$ (continued) $\boxed{\text{solved with the three partial fractions}}$

$\boxed{\text{part 1}}$ $\mathcal{L}^{-1}\left\{\frac{6}{s(s-3)(s-4)}\right\} = \mathcal{L}^{-1}\left\{\frac{A}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{B}{s-3}\right\} + \mathcal{L}^{-1}\left\{\frac{C}{s-4}\right\}$

$$\boxed{= A(1) + Be^{3t} + Ce^{4t}}$$

$A(s-3)(s-4) + B(s)(s-4) + C(s)(s-3) = 6$

$A(s^2-7s+12) + B(s^2-4s) + C(s^2-3s) = 6$

$As^2 - 7A\cdot s + 12A + Bs^2 - 4B\cdot s + Cs^2 - 3C\cdot s = 6$

$s^2(A+B+C) + s(-7A-3C-4B) + 12A = 6$

$12A = 6$ $\qquad B + C = -\frac{1}{2}$ $\qquad 3B + \cancel{3C} = -\frac{3}{2}$ $\qquad B + C = -\frac{1}{2}$

$\boxed{A = \frac{1}{2}}$ $\qquad -4B - 3C = \frac{7}{2}$ $\qquad -4B - \cancel{3C} = \frac{7}{2}$ $\qquad -2 + C = -\frac{1}{2}$

$-B = \frac{4}{2} = 2$ $\qquad \boxed{C = \frac{3}{2}}$

$\boxed{B = -2}$

$\boxed{\text{part 1 solution}}$

$$\boxed{\frac{1}{2} - 2e^{3t} + \frac{3}{2}e^{4t}}$$

$\boxed{\text{part 2}}$ $\mathcal{L}^{-1}\left\{\frac{-3}{s(s+1)(s-4)}\right\} = \mathcal{L}^{-1}\left\{\frac{A}{s} + \frac{B}{s+1} + \frac{C}{s-4}\right\}$

$$= \boxed{A(1) + Be^{-t} + Ce^{4t}}$$

$A(s+1)(s-4) + B(s)(s-4) + C(s)(s+1) = -3$

$A(s^2-3s-4) + B(s^2-4s) + C(s^2+s) = -3$

$As^2 - 3A\cdot s - 4A + Bs^2 - 4B\cdot s + Cs^2 + C\cdot s = -3$

$s^2(A+B+C) + s(-3A-4B+C) - 4A = -3$

$\boxed{\text{part 2}}$ continued

$-4A = -3 \qquad A + B + C = 0 \qquad -3A - 4B + C = 0$

$\boxed{A = +\frac{3}{4}} \qquad B + C = -\frac{3}{4} \qquad -3\left(+\frac{3}{4}\right) - 4B + C = 0$

$-4B + C = +\frac{9}{4}$

$$-B - \cancel{C} = \frac{3}{4}$$
$$-4B + \cancel{C} = \frac{9}{4}$$
$$-5B = \frac{12}{4} = 3$$
$$\boxed{B = -\frac{3}{5}}$$

$$B + C = -\frac{3}{4}$$
$$-\frac{3}{5} + C = -\frac{3}{4}$$
$$C = -\frac{3}{4} + \frac{3}{5} = \frac{-15+12}{20} = \boxed{\frac{-3}{20} = C}$$

$\boxed{\text{part 2 solution}}$

$$\boxed{\frac{3}{4} + \frac{-3}{5}e^{-t} - \frac{3}{20}e^{4t}}$$

$\boxed{\text{part 3}}$

$$\mathcal{L}^{-1}\left\{\frac{s-5}{(s-4)s}\right\} = \mathcal{L}^{-1}\left\{\frac{A}{s} + \frac{B}{s-4}\right\}$$

$$\boxed{A(1) + Be^{4t}}$$

$$A(s-4) + B(s) = s-5$$
$$As - 4A + Bs = s-5$$
$$s(A+B) - 4A = s-5$$

$$-4A = -5$$
$$\boxed{A = +5/4}$$

$$A + B = 1$$
$$+5/4 + B = 1$$
$$\boxed{B = -1/4}$$

$\boxed{\text{part 3 solution}}$

$$\boxed{\frac{+5}{4} - \frac{1}{4}e^{4t}}$$

$\boxed{\text{total solution}}$

$$\cancel{\frac{1}{2}} - 2e^{3t} + \frac{3}{2}e^{4t} + \cancel{\frac{3}{4}} - \frac{3}{5}e^{-t} - \frac{3}{20}e^{4t} + \cancel{\frac{5}{4}} - \frac{1}{4}e^{4t}$$

$$\frac{5}{2} - 2e^{3t} - \frac{3}{5}e^{-t} + \left(\frac{3}{2} - \frac{3}{20} - \frac{1}{4}\right)e^{4t} = \boxed{-2e^{3t} - \frac{3}{5}e^{-t} + \frac{5}{2} + \frac{11}{10}e^{4t} = y}$$

$$\frac{30 - 3 - 5}{20} = \frac{22}{20} = \frac{11}{10}$$

practice problems for transforms of derivatives

① $y'' + 11y' + 24y = 0 \quad y(0) = -1 \quad y'(0) = 0$

② $y'' + 3y' - 10y = 0 \quad y(0) = -1 \quad y'(0) = 1$

③ $y'' - y' = e^t \quad y(0) = 0 \quad y'(0) = 1$

④ $y''' + y' = e^t \quad y(0) = y'(0) = y''(0) = 0$

⑤ $y^{(4)} + y'' = 1 \quad y(0) = y'(0) = y''(0) = y'''(0) = 0$

⑥ $y'' + 2y' + y = e^{-t} \quad y(0) = y'(0) = 0$

⑦ $y''' - y' = e^t + e^{-t} \quad y(0) = y'(0) = y''(0) = 0$

⑧ $y'' - 4y' = 0 \quad y(0) = 3 \quad y'(0) = 8$

answers

① $y = \frac{1}{5} e^{-8t}(3 - 8e^{5t})$

② $y = -\frac{1}{7} e^{-5t}(3 + 4e^{7t})$

③ $y = te^t$

④ $y = \frac{1}{2}(-2 + e^t + \cos t - \sin t)$

⑤ $y = \frac{1}{2}(2\cos t + t^2 - 2)$

⑥ $y = t^2 e^{-t}$

⑦ $y = \frac{1}{2} e^{-t}(1 - e^{2t} + t + te^{2t})$

⑧ $y = 1 + 2e^{4t}$