Section 7.3

Operational Properties of LaPlace Transforms

First translational Theorem (on s-axis) $\mathcal{L}\{e^{at}f(t)\} = F(s-a)$

2nd translational Theorem (on t-axis) $\mathcal{L}\{f(t-a)\,\mathcal{U}(t-a)\} = e^{-as}F(s)$

where $\mathcal{U}(t-a) = \begin{cases} 0 & 0 \le t < a \\ 1 & t \ge a \end{cases}$

$\mathcal{L}\{\mathcal{U}(t-a)\} = \dfrac{e^{-as}}{s}$ $\qquad$ $\mathcal{L}\{g(t)\,\mathcal{U}(t-a)\} = e^{-as}\mathcal{L}\{g(t+a)\}$.

Example Problems First translational Theorem

Example 1 $\mathcal{L}\{t^5 e^{-3t}\}$

this uses $\mathcal{L}\{e^{at}f(t)\} = F(s-a)$ shift on the s-axis

or $\mathcal{L}\{e^{at}f(t)\} = F(s)\big|_{s\to s-a}$

we have $f(t) = t^5$

so $\mathcal{L}\{t^5 e^{-3t}\} = F(s+3)$ $\qquad$ goal: Find $F(s)$ then account for the shift

$f(t) = t^5$ $\qquad \left[\mathcal{L}\{t^n\} = \dfrac{n!}{s^{n+1}}\right]$

$\mathcal{L}\{t^5\} = \dfrac{5!}{s^{5+1}} = \dfrac{5!}{s^6}$ $\quad\Big|\quad$ $\mathcal{L}\{t^5 e^{-3t}\} = \boxed{\dfrac{5!}{(s+3)^6}}$

$\boxed{\text{Let's try this in reverse}}$

$\boxed{\text{Example 4}}$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-4)^3}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^3}\Big|_{s\to s-4}\right\}$$

For $\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\}$, we need $\mathcal{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n$

$n+1=3$

$n=2$

$$\boxed{\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\}} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^3}\right\} = \boxed{\frac{1}{2}t^2}$$

So $\mathcal{L}^{-1}\left\{\frac{1}{(s-4)^3}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^3}\Big|_{\underbrace{s\to s-4}_{e^{4t}}}\right\} = \boxed{\frac{1}{2}t^2e^{4t}}$

$\boxed{\text{Example 5}}$ $\boxed{\mathcal{L}^{-1}\left\{\frac{5}{s^2-6s+10}\right\}}$

$$5\mathcal{L}^{-1}\left\{\frac{1}{s^2-6s+9+1}\right\} = 5\mathcal{L}^{-1}\left\{\frac{1}{(s-3)^2+1}\right\}$$

a form of $\frac{k}{s^2+k^2}$

$\Rightarrow \sin kt$ with $k=1$

$$= 5\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\Big|_{\underbrace{s\to s-3}_{e^{3t}}}\right\}$$

$$= 5\left(e^{3t}\sin t\right) = \boxed{5e^{3t}\sin t}$$

Example 6 $\quad \mathcal{L}^{-1}\left\{ \dfrac{s}{(s-4)^2} \right\}$

For this problem, we need to rearrange it because in its present form, it doesn't match any of our elementary LaPlace transforms.

$R = s-4$
$R+4 = s$

$\mathcal{L}\left\{ \dfrac{R+4}{R^2} \right\}$

$= \mathcal{L}^{-1}\left\{ \dfrac{R}{R^2} \right\} + \mathcal{L}^{-1}\left\{ \dfrac{4}{R^2} \right\}$

$= \mathcal{L}^{-1}\left\{ \dfrac{1}{R} \right\} + 4\,\mathcal{L}^{-1}\left\{ \dfrac{1}{R^2} \right\}$

$n+1=1 \qquad n+1=2$
$n=0 \qquad n=1$

$= \underbrace{\quad}_{1} \qquad \underbrace{\quad}_{4t}$

Reference transform
$\mathcal{L}\{t^n\} = \dfrac{n!}{s^{n+1}}$

Recall the initial shift $\quad s-4 \rightarrow$ so we need e^{4t}

final answer

$$\boxed{(1+4t)e^{4t}}$$

Example 7

$$\boxed{\text{Solve } y'' - 2y' + 5y = 1 + t \quad y(0) = 0,\ y'(0) = 4}$$

$$\mathcal{L}\{y''\} - 2\mathcal{L}\{y'\} + 5\mathcal{L}\{y\} = \mathcal{L}\{1+t\}$$

$$s^2F(s) - \cancel{s\,y(0)}^{\,0} - \cancel{y'(0)}^{\,4} - 2[sF(s) - \cancel{y(0)}^{\,0}] + 5\mathcal{L}\{y\}$$
$$= \mathcal{L}\{1\} + \mathcal{L}\{t\}.$$

$$s^2\mathcal{L}\{y\} - 4 - 2s\mathcal{L}\{y\} + 5\mathcal{L}\{y\} = \frac{1}{s} + \frac{1}{s^2}$$

$$\mathcal{L}\{y\}(s^2 - 2s + 5) = \frac{1}{s} + \frac{1}{s^2} + 4$$

$$\mathcal{L}\{y\} = \left(\frac{1}{s} + \frac{1}{s^2} + 4\right)\frac{1}{\underbrace{(s^2-2s+5)}_{\substack{(s^2-2s+1+4) \\ (s-1)^2+4}}}$$

$$y = \mathcal{L}^{-1}\left\{\left(\frac{1}{s} + \frac{1}{s^2} + 4\right)\left(\frac{1}{(s-1)^2+4}\right)\right\}$$

$$y = \mathcal{L}^{-1}\left\{\frac{1}{s}\cdot\frac{1}{(s-1)^2+4}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2}\cdot\frac{1}{(s-1)^2+4}\right\}$$
$$+ \mathcal{L}^{-1}\left\{4\left(\frac{1}{(s-1)^2+4}\right)\right\}$$

Let's treat this as a partial fractions problem

$$\left(\frac{1}{s} + \frac{1}{s^2} + 4\right) = \left(\frac{s + 1 + 4s^2}{s^2}\right)$$

so we have

$$y = \mathcal{L}^{-1}\left\{\frac{(4s^2+s+1)}{s^2(s^2-2s+5)}\right\}$$

$$y = \mathcal{L}^{-1}\left\{\frac{A}{s^2} + \frac{B}{s} + \frac{Cs+D}{(s^2-2s+5)}\right\}$$

$$y = \mathcal{L}^{-1}\left\{\frac{A}{s^2} + \frac{B}{s} + \frac{Cs}{(s-1)^2+4} + \frac{D}{(s-1)^2+4}\right\}$$

$n+1=2$	$n+1=1$	$k=2$	$k=2$
$n=1$	$n=0$	$\cos kt$	$\sin kt$
		with e^t shift	with e^t shift

$$\boxed{\begin{aligned} y &= At + B + Ce^t\cos 2t + \tfrac{1}{2}De^t\sin 2t \\ y &= \tfrac{1}{5}t + \tfrac{7}{25} - \tfrac{7}{25}e^t\cos 2t + \tfrac{109}{50}e^t\sin 2t \end{aligned}}$$

now find the coefficients

$$A(s^2-2s+5)+B(s)(s^2-2s+5) + (Cs+D)(s^2) = 4s^2+s+1$$

$A(s^2-2s+5)$

$B(s^3-2s^2+5s)$

$C(s^3)$

$D(s^2)$

$s^2(A-2B+D) = 4s^2$

$s^3(B+C) = 0$

$s(-2A+5B) = 1$

$(5A) = 1$

$5A=1$, $\boxed{A=\frac{1}{5}}$

$-2A+5B=1$

$-2(\frac{1}{5})+5B=1$

$5B=\frac{7}{5}$

$\boxed{B=\frac{7}{25}}$

$B+C=0$

$\frac{7}{25}+C=0$

$\boxed{C=\frac{-7}{25}}$

$A-2B+D=4$

$\frac{5}{25}-2\left(\frac{7}{25}\right)+D=\frac{100}{25}$

$\boxed{D=\frac{109}{25}}$

Practice Exercises For the first translational theorem

Apply the 1st translational theorem to find the LaPlace transforms.

① $f(t) = t^4 e^{\pi t}$

② $f(t) = e^{-2t}\sin 3\pi t$

③ $f(t) = t^{3/2} e^{-4t}$

④ $f(t) = e^{-t/2}\cos(t - \frac{1}{8}\pi)$

Apply the first translational theorem to find the inverse LaPlace Transforms of each function.

⑤ $F(s) = \dfrac{3}{2s-4}$

⑥ $F(s) = \dfrac{s-1}{(s+1)^3}$

⑦ $F(s) = \dfrac{1}{s^2+4s+4}$

⑧ $F(s) = \dfrac{s+2}{s^2+4s+5}$

⑨ $F(s) = \dfrac{3s+5}{s^2-6s+25}$

⑩ $F(s) = \dfrac{2s-3}{9s^2-12s+20}$

Use partial fractions to find the inverse Laplace transforms of each function

⑪ $F(s) = \dfrac{1}{s^2-4}$

⑫ $F(s) = \dfrac{5-2s}{s^2+7s+10}$

⑬ $F(s) = \dfrac{1}{s^3-5s^2}$

⑭ $F(s) = \dfrac{1}{s^4-16}$

⑮ $F(s) = \dfrac{s^2-2s}{s^4+5s^2+4}$

⑯ $F(s) = \dfrac{s^2+3}{(s^2+2s+2)^2}$

Use Laplace transforms to solve each initial value problem

17. $y'' + 6y' + 25y = 0 \qquad y(0) = 2,\ y'(0) = 3$

18. $y'' - 4y = 3t \qquad y(0) = y'(0) = 0$

19. $x^{(3)} + x'' - 6x' = 0 \qquad x(0) = 0,\ x'(0) = x''(0) = 1$

20. $x^{(4)} + x = 0 \qquad x(0) = x'(0) = x''(0),\ x^{(3)}(0) = 1$

21. $x'' + 4x' + 13x = te^{-t} \qquad x(0) = 0,\ x'(0) = 2$

Answers

1. $24/(s-\pi)^5$
2. $\dfrac{3\pi}{[(s+2)^2 + 9\pi^2]}$
3. $\frac{3}{4}\sqrt{\pi}\,(s+4)^{-5/2}$
4. $\dfrac{\sqrt{2}\,(2s+5)}{(4s^2+4s+17)}$
5. $\frac{3}{2}e^{2t}$
6. $(t - t^2)e^{-t}$
7. te^{-2t}
8. $e^{-2t}\cos t$
9. $e^{3t}(3\cos 4t + \frac{7}{2}\sin 4t)$
10. $\frac{1}{36}e^{2t/3}(8\cos\frac{4}{3}t - 5\sin\frac{4}{3}t)$
11. $\frac{1}{2}\sinh 2t$
12. $3e^{-2t} - 5e^{-5t}$
13. $\frac{1}{25}(e^{5t} - 1 - 5t)$
14. $\frac{1}{16}(\sinh 2t - \sin 2t)$
15. $\frac{1}{3}(2\cos 2t + 2\sin 2t - 2\cos t - \sin t)$
16. $\frac{1}{2}e^{-t}(5\sin t - 3t\cos t - 2t\sin t)$
17. $\frac{1}{4}e^{-3t}(8\cos 4t + 9\sin 4t)$
18. $\frac{1}{8}(-6t + 3\sinh 2t)$
19. $\frac{1}{15}(6e^{2t} - 5 - e^{-3t})$

$r = 1/\sqrt{2}$

20. $x(t) = r(\cosh rt \sin rt - \sinh rt \cos rt$
21. $\frac{1}{50}[(5t-1)e^{-t} + e^{-2t}(\cos 3t + 32\sin 3t)]$