Foundations of Discrete Mathematics

Chapter 0

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Statement

- Statement is an ordinary English statement of fact.
- It has a subject, a verb, and a predicate.
- It can be assigned a "true value," which can be classified as being either **true** or **false**.

Examples of Statement

- "There are 168 primes less than 1000." ← **True**
- "Seventeen is an even number." ← **False**
- "$\sqrt{3}^{\sqrt{3}}$ is a rational number." ← **False**
- "Zero is not negative." ← **True**

Compound Statements

- A compound statement is a statement formed from two other statements.
- These both statements can be linked with “and” or “or.”

A Compound Statement with “and”

“9 = 32 and 3.14 < π”

- This compound statement is formed from two simple statements:
  - **“9 = 32” and**
  - **“3.14 < π”**

Rule for a Compound Statement with “and”

- Given the statements p and q.
- The compound statement “p and q” is true <u>if both p and q are true</u>.
- “p and q” is false <u>if either p is false or q is false.</u>

Examples of Compound Statements with "and"

- "$-2^2 = -4$ and $5 < 100$" ← **True**
- "$-2^2 + 32 = 42$ and $3.14 < \pi$"

↑ **False**

A Compound Statement with "or"

"The man is wanted dead or alive"

- This compound statement is formed from two simple statements:
  - **"The man is wanted dead"**
  - **"alive"**

A Compound Statement with "or"

- There are
  - **Inclusive OR**
  - **Exclusive OR**

- In this course, we will use Inclusive OR, which includes the possibility of both p and q statements.

Rule for a Compound Statement with “or”

- Given the statements p and q.
- The compound statement “p or q” is true <u>if p is true or q is true or both are true.</u>
- “p or q” is false only <u>when p and q are false.</u>

Examples of Compound Statements with "or"

- "7 + 5 = 12 or 571 is the 125th prime." ←**True**

- "5 is an even number or √8 > 3."

↑ **False**

Implication

- Statement of the form “p implies q”
- Where p and q are statements.
  - p is called hypothesis.
  - q is called conclusion.
- The symbol $\rightarrow$ is read ***implies***.

Examples of Implication

- "2 is an even integer $\rightarrow$ 4 is an even integer"

↑ **Hypothesis**

↑ **Conclusion**

Implication

- Implications often appears without the word implies.

"2 is an even integer, then 4 is an even integer."

Implication

- The implication "$p \rightarrow q$" is false only when
  - the hypothesis **p is true** and
  - the conclusion **q is false**.

- In all other situations, **it is true**.

Implication

- "If -1 is a positive number, then 2+2=5."

↑ **True**

Why?

"If -1 is a positive number" ← **False**

"2+2=5" ← **False**

Implication

- “If -1 is a positive number, then 2+2=4.”

↑ **True**

Why?

“If -1 is a positive number” ←**False**

“2+2=5” ← **True**

The Converse of an Implication

- The converse of the implication $p \rightarrow q$ is the implication $q \rightarrow p$.

Given the implication

“2 is an even integer, then 4 is an even integer.”

The Converse of an Implication

- The converse of the implication $p \rightarrow q$ is the implication $q \rightarrow p$.

The converse is

"2 is an even integer, then 4 is an even integer."

The Converse of an Implication

Given the implication

"If $4^2 = 16$, then $-1^2 = 1$"

The converse is

"If $-1^2 = 1$, then $4^2 = 16$"

Double Implication

- The double implication $p \leftrightarrow q$ is read "p if and only if q."

"$p \rightarrow q$" and "$p \leftarrow q$" or

"$p \rightarrow q$" and "$q \rightarrow p$"

Double Implication

- The double implication “$p \leftrightarrow q$” is true if p and q have the same truth values;
- “$p \leftrightarrow q$” is false if p and q have different truth values.

Examples of Double Implication

- "2 is an even number $\leftrightarrow$ 4 is an even number"

  $\uparrow$ **True**

"2 is an even number" $\leftarrow$ **True**

"4 is an even number" $\leftarrow$ **True**

Examples of Double Implication

- "2 is an even number $\leftrightarrow$ 5 is an even number"

  $\uparrow$ **False**

"2 is an even number" $\leftarrow$ **True**

"5 is an even number" $\leftarrow$ **False**

Is this Double Implication True or False?

1. "$4^2 = 16 \leftrightarrow -1^2 = -1$"

↑ **True**

"$4^2 = 16$" ← **True**

"$-1^2 = -1$" ← **True**

- **Both statements are true.**

Is this Double Implication True or False?

2. "$4^2 = 16$ if an only if $(-1)^2 = -1$"

↑ **False**

"$4^2 = 16$" ← **True**

"$-1^2 = -1$" ← **False**

- **The two statements have different truth values.**

Is this Double Implication True or False?

3. "$4^2 = 15$ if an only if $-1^2 = -1$"

↑ **False**

"$4^2 = 15$" ← **False**

"$-1^2 = -1$" ← **True**

- **The two statements have different truth values.**

Is this Double Implication True or False?

4. "$4^2 = 15 \leftrightarrow (-1)^2 = -1$"

↑ **True**

"$4^2 = 16$" ← **False**

"$(-1)^2 = -1$" ← **False**

- **Both statements are false.**

Negation

- The negation of the statement p is the statement that asserts that p is not true.
- The negation of p is denoted by ¬p (“not p”).

Example of Negation

- The statement “x equals to 4” (“$x = 4$”)
- The negation is “x does not equal to 4” (“$x \neq 4$”)
- $\neq$ means “not equal.”

Negation

- "not p" can be expressed as

  "It is not the case that p."

- "25 is a perfect square."

- "25 is not a perfect square."

Negation

- "not p" can be expressed as

  "It is not the case that p."

- "25 is a perfect square."

- "It is not the case that 25 is a perfect square."

Negation

- The negation of an “or” statement is always an “and” statement.
- The negation of an “and” statement is always an “or.”

Negation

- The negation of "p and q" is the assertion "$\neg p$ or $\neg q$."

- The negation of
  "$a^2 + b^2 = c^2$ and $a > 0$" is

  "Either $a^2 + b^2 \neq c^2$ or $a \leq 0$."

Negation

- The negation of "p or q" is the assertion "¬p and ¬q."

- The negation of

"x + y = 6 or 2x + 3y < 7" is

"x + y ≠ 6 and 2x + 3y ≥ 7."

Negation

- What is the negation of $p \to q$?
- "Not $p \to q$" means $p \to q$ is false because p is true and q is false.
- **$\neg(p \to q)$** is **"p and $\neg q$"**

The Contrapositive

- The contrapositive of the implication "$p \rightarrow q$" is the implication "$(\neg q) \rightarrow (\neg p)$."

Examples of the Contrapositive

- "If x is an even number, then $x^2 + 3x$ is an even number"

  The contrapositive is

- "If $x^2 + 3x$ is an odd number, then x is an odd number. "

Examples of the Contrapositive

- "If $4^2 = 16$, then $-1^2 = 1$"

The contrapositive is

- "If $-1^2 \neq 1$, then $4^2 \neq 16$."

↑ **is false because the hypothesis is true and the conclusion is false.**

Examples of the Contrapositive

- "If $-1^2 = 1$, then $4^2 = 16$"

The contrapositive is

- "If $4^2 \neq 16$, then $-1^2 \neq 1$."

↑ **is true because the hypothesis is false and the conclusion is true.**

Quantifiers

- The expressions ***there exits*** and ***for all*** are quantifiers.
- ***"for any"*** and ***"all"*** are synonymous with ***"for all."***

Quantifiers

- The universal quantifier ***for all*** says that
- a statement is true ***for all*** integers or ***for all*** polynomials or ***for all*** elements of a certain type.

Quantifiers

- The existential quantifier ***there exists*** stipulates the existence of a single element for which a statement is true.

Examples of Quantifiers

- $x^2 + x + 1 > 0$ for all real numbers x.
- All polynomials are continuous functions.
- For all real numbers $x > 0$, x has a real square root.

Examples of Quantifiers

- For any positive integer n, 2(1 + 2 + 3 + … + n) = n x (n + 1).
- (AB)C = A(BC) for all square matrices, A, B, and C.

Examples of Quantifiers

- Some polynomial have no real zeros.
- “There exists a set A and a set B such that A and B have no element in common.”

Examples of Quantifiers

- There exists a smallest positive integer.
- Two sets may have no element in common.

Quantifiers

- Rewrite “Some polynomials have no real zeros” making use of the existential quantifier.

- **There exists a polynomial with no real zeros.**

Quantifiers

- ***There exists*** a matrix 0 with the property that A + 0 = 0 + A ***for all*** matrices A.

- ***For any*** real number x, ***there exists*** an integer such that $n \le x < n + 1$

Quantifiers

- ***Every*** positive integer is the product of primes.
- ***Every*** nonempty set of positive integers has a smallest element.

To Negate Quantifiers

- To negate a statement that involves one or more quantifiers in a useful way can be difficult.

- In this situation begin with ***"It is not the case"*** *and then to reflect on what you have written.*

To Negate Quantifiers

- "For every real number x, x has a real square root."
- ***"It is not the case*** *that every real number x has a real square root."*

or

- ***"There exists*** *a real number that does not have a real square root."*

To Negate Quantifiers

- The negation of ***"For all something, p"*** is the statement ***"There exists something such that ¬p."***

- The negation of ***"There exists something such that p"*** is the statement ***"For all something, ¬p."***

To Negate Quantifiers

- The negation of

"There exists a and b for which ab ≠ ba",

is the statement

"For all a and b, ab = ba."

The Symbols $\forall$ and $\exists$

- The symbols $\forall$ and $\exists$ are commonly used for the quantifiers for all and there exists, respectively.

$$\forall x,\ \exists\, n \text{ such that } n > x$$

or

$$\forall x,\ \exists\, n,\ n > x$$

Some Assumptions

- ***The product of nonzero real numbers is nonzero.***
- ***The square of a nonzero real number is a positive real number.***
- ***A prime is a positive integer $p > 1$ that is divisible evenly only by ± 1 and $\pm p$.***

Some Assumptions

- ***An even integer is one that is of the form 2k for some integer k.***
- ***An odd integer is one that is of the form 2k + 1 for some integer k.***
- ***The product of two even integers is even.***

Some Assumptions

- ***The product of two odd integers is odd.***
- ***The product of an odd integer and an even integer is even.***
- ***A real number is rational if it is a common fraction, that is, the quotient m/n of the integers m and n with n≠0.***

Some Assumptions

- ***A real number is irrational if it is not rational. For example π and $\sqrt[3]{5}$***
- ***An irrational number has a decimal expansion that neither repeats nor terminates.***

Proofs in Mathematics

- Many mathematical theorems are statements that a certain implication is true.

- The hypothesis and conclusion of an implication could be any two statements, even statements completely unrelated to each other.

Proofs in Mathematics

- Suppose

$$\text{“}0 < x < 1 \rightarrow x^2 < 1\text{”}$$

- To prove this statement a general argument must be given that works for all x between 0 and 1.

Proofs in Mathematics

- Assume that the hypothesis is true.

  $0 < x < 1$ **← Hypothesis is true**

- x is a real number with $0 < x < 1$

  - $x > 0$ and $x < 1$

Proofs in Mathematics

- Multiplying $x < 1$ by a positive such as x preserves the inequality.

$$x \cdot x < x \cdot 1$$

$$x^2 < x$$

Since $x < 1$, $x^2 < 1$

↑ **This argument works for all x between 0 and 1**

Proofs in Mathematics

- Suppose

"$x^2 < 1 \rightarrow 0 < x < 1$ " $\leftarrow$ **False**

- When $x = -1/2$

$(-1/2)^2 < 1$ $\leftarrow$ **Left side is true**

$0 < -1/2 < 1$ $\leftarrow$ **Right side is False**

Proofs in Mathematics

- To show that a theorem, or a step in a proof, is false, it is enough to find a single case where the implication does not hold.

- To show that a theorem is true, we must give a proof that covers all possible cases.

Proofs in Mathematics

- Is the contrapositive of the following statement true?

"$x^2 \geq 1 \rightarrow (x \leq 0 \text{ or } x \geq 1)$"

Proofs in Mathematics

“$x^2 + y^2 = 0 \leftrightarrow (x = 0 \text{ and } y = 0)$”

- This statement is of type $A \leftrightarrow B$
- It can be expressed as “A is a necessary and sufficient condition for B”

Proofs in Mathematics

- The statement is of type

$$(A \text{ and } B) \leftrightarrow C$$

- It can be expressed as "A and B are necessary and sufficient conditions for C."

Proofs in Mathematics

- ***"A triangle has three equals angles"*** is a necessary and sufficient condition for ***"a triangle has three equal sides."***

- To prove that "A $\leftrightarrow$ B" is true, we must prove separately that "A $\rightarrow$ B" and "B $\rightarrow$ A" are both true.

Proofs in Mathematics

- Prove that

  $$\text{“}x^2 + y^2 = 0 \leftrightarrow (x = 0 \text{ and } y = 0)\text{”}$$

- Assume $x^2 + y^2 = 0$

- Since ***the square of a real number cannot be negative*** and ***the square of a nonzero real number is positive*.**

Proofs in Mathematics

- If either "$x^2 \neq 0$ or $y^2 \neq 0$,

the sum $x^2 + y^2$ would be positive, which is not true.

- This means $x^2 = 0$ and $y^2 = 0$, so $x = 0$ and $y = 0$, as desired.

Proofs in Mathematics

- A theorem in mathematics asserts that three or more statements are equivalent, meaning that all possible implications between pairs of statements are true.

Proofs in Mathematics

- "The following are equivalent:
  1. A
  2. B
  3. C"

- This means *that each of the double implications* $A \leftrightarrow B$*,* $B \leftrightarrow C$*,* $A \leftrightarrow C$ *is true.*

Proofs in Mathematics

$$A \leftrightarrow B,\ B \leftrightarrow C,\ A \leftrightarrow C \textit{ is true.}$$

- Instead of proving the truth of the six implications, it is more efficient just to establish the truth of the sequence

$$A \rightarrow B \rightarrow C$$

Example: Proofs in Mathematics

- Let x be a real number. Show that the following are equivalent.

1. $x = \pm 1$.
2. $x^2 = 1$.
3. If a is any real number, then $ax = \pm a$

Example: Proofs in Mathematics

- It is sufficient to establish the truth of the sequence

$$(2) \rightarrow (1) \rightarrow (3) \rightarrow (2)$$

$(x^2 = 1) \rightarrow (x = \pm 1)$

$(x = \pm 1) \rightarrow$ (If a is any real number, then $ax = \pm a$)

(If a is any real number, then $ax = \pm a$) $\rightarrow (x^2 = 1)$

Example: Proofs in Mathematics

- ***(2) → (1)*** ← Assume (2) and prove (1)

Since

$$x^2 = 1,\ 0 = x^2 - 1 = (x + 1)(x - 1)$$

- Since the product of real numbers is zero if and only if one of the numbers is zero.

Example: Proofs in Mathematics

- *(2)* → *(1)*   ← Assume (2) and prove (1)

Either

$x + 1 = 0$ or $x - 1 = 0$

- Hence x = -1 or x = +1, as required.

Example: Proofs in Mathematics

- *(1)* → *(3)*   ← Assume (1) and prove (3)

Either $x = +1$ or $x = -1$

- Let a be a real number. If $x = +1$, then $ax = a \cdot 1 = a$.
- If $x = -1$, then $ax = -a$
- In every case, $ax = \pm a$ as required.

Example: Proofs in Mathematics

- ***(3) → (2)*** ← Assume (3) and prove (2)

Given that

$ax = \pm a$ for any real number a.

- With $a = 1$, we obtain $x = \pm 1$ and squaring gives $x^2 = 1$, as desired.

Direct Proofs

- Most theorems in mathematics are stated as implications: A $\rightarrow$ B.
- Sometimes, it is possible to prove such a statement directly.
- By establishing the validity of a sequence of implications.

Prove that for all real numbers x, $x^2 - 4x + 17 \neq 0$

The left side of the inequality can be represented as

- $x^2 - 4x + 17 = x^2 - 4x + 4 + 13$

  $= (x - 2)^2 + 13$

Prove that for all real numbers x, $x^2 - 4x + 17 \neq 0$

- $(x - 2)^2 + 13$ is the sum of 13 and a number.
- $(x - 2)^2$ is never negative
- So, $x^2 - 4x + 17 \geq 13$ for any x;
- In particular $x^2 - 4x + 17 \neq 0$

Suppose that x and y are real numbers such that $2x + y = 1$ and $x - y = -4$

Prove that $x = -1$ and $y = 3$

- $(2x + y = 1 \text{ and } x - y = -4) \rightarrow$

  $(2x + y) + (x - y) = 1 - 4$

- $2x + y + x - y = 1 - 4$

  $\rightarrow 3x = -3 \rightarrow x = -1$

Suppose that x and y are real numbers such that 2x + y = 1 and x − y = -4

Also,

- $(x = -1 \text{ and } x - y = -4) \rightarrow$
- $(-1 - y = -4) \rightarrow -y = -1 + 4 = -3.$

  $\rightarrow y = -3.$

Proof by Cases

- A direct argument is made simpler by breaking it into a number of cases, one of which must hold and each of which leads to the desired conclusion.

Example: Proof by Cases

Let n be an integer.

Prove that $9n^2 + 3n - 2$ is even.

Case 1. n is even

1. An integer is even if and only if twice another integer.
2. $n = 2k$ for some integer k.
3. Thus $9n^2 + 3n - 2 = 36k^2 + 6k - 2$

$$= 2(18k^2 + 3k - 1)$$

↑**Even**

Example: Proof by Cases

Case 2. n is odd.

1. An integer is odd if and only if it has the form $2k + 1$ for some integer k.
2. $n = 2k + 1$ for some integer k.
3. Thus $9n^2 + 3n - 2$

$= 9(4k^2 + 4k + 1) + 3(2k + 1) - 2$

$= 36k^2 + 42k + 10$

$= 2(18k^2 + 21k + 5)$ ← **Even**

Prove the Contrapositive

- “A $\rightarrow$ B” is true if an only if its contrapositive “$\neg$A $\rightarrow$ $\neg$B” is true.
- “A $\rightarrow$ B” is false if and only if A is true and B is false.
- That is, if and only if “$\neg$B $\rightarrow$ $\neg$A” is false.

Prove the Contrapositive

- Two statements
  “$A \to B$” and “$\neg B \to \neg A$” are false together (or true together).

- They have the same true values. The result is proved.

Prove the Contrapositive

- If the average of four different integers is 10, prove that one of the integers is greater than 11.

Let A and B the statements

A: ” The average of four integers, all different, is 10.”

B: “One of the four integers is greater than 11.”

Prove the Contrapositive

- We will prove the truth of "A $\rightarrow$B" proving the truth of the contrapositive "$\neg$B $\rightarrow$ $\neg$A."

Using the theorem

"A $\rightarrow$ B" is true if an only if its contrapositive "$\neg$B $\rightarrow$ $\neg$A" is true.

Prove the Contrapositive

- Call the given integers a, b, c, d
- If B is false, then each of these numbers is at most 11.

Prove the Contrapositive

- Since they are all different,

  the biggest value for a + b + c + d is 11 + 10 + 9 +8 = 38.

  So the biggest possible average would be 38/4, which is less than 10, so A is false.

Prove by Contradiction

- Assuming that the negation of the statement A is true.
- If this assumption leads to a statement that is obviously false (an absurdity) or to a statement that contradicts something else, then $\neg A$ is false.
- So, A must be true.

Show that there is no largest integer

- Let A be the statement "There is no largest number."
- If A is false, then there is a largest integer N.
- This is absurd, however, because N+1 is an integer larger than N. Thus $\neg A$ is false. So, A is true.

Example

- Suppose that a is nonzero rational number and that b is an irrational number. Prove that ab is irrational.

- By contradiction assume

  A: ab is irrational is false, then ab is rational, so $ab = m / n$ for integers m and n, $n \neq 0$.

Example

- Now a is given to be rational, so $a = k/l$ for integers k and l, $l \neq 0$, and $k \neq 0$ (because $a \neq 0$).

- $b = m / na = ml / nk$ $(ab = m/n)$

  with $nk \neq 0$, so b is rational

  ↑ **This is not true**

By Contradiction , we have proven that A is true

Prove that $\sqrt{2}$ is an irrational number

- If the statement is false, then there exist integers m and n such that $\sqrt{2}=m/n$.

- If both m and n are even, we can cancel 2’s in numerator and denominator until at least one of them is odd.

Prove that $\sqrt{2}$ is an irrational number

- Without loss of generality, we may assume that not both m and n are even.
  - Squaring both sides of $\sqrt{2} = m/n$
  - $2 = m^2 / n^2$
  - $m^2 = 2n^2$, so m^2 is even.

Prove that √2 is an irrational number

- The square of an odd integer is odd,
  - $m = 2k$ must be even
  - $m^2 = 2n^2$
  - $4k^2 = 2n^2$
  - $2k^2 = n^2$.

↑ **It implies that n is even, contradicting the fact that not both m and n are even.**

Topics covered in this Meeting

- Compound statements
  - And and Or
  - Implication and its converse
  - The Contrapositive
  - Quantifiers
  - Negation

Topics covered in this Meeting

- Proofs in Mathematics
  - Direct Proof
  - Proof by cases
  - Proof the contrapositive
  - Proof by contradiction

Reference

- "Discrete Mathematics with Graph Theory", Third Edition, E. Goodaire and Michael Parmenter, Pearson Prentice Hall, 2006. pp 1-18.