Foundations of Discrete Mathematics

Chapters 5

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Mathematical Induction

- Mathematical induction is one of the most basic methods of proof.
- It is applied in every area of mathematics.

Mathematical Induction

- Mathematical induction is used to prove propositions of the form

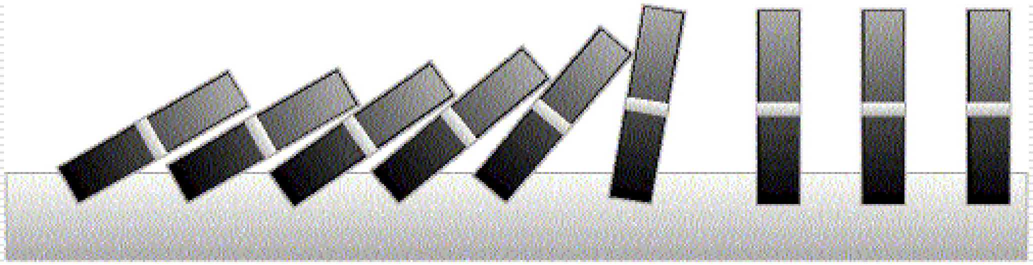
$$\forall n \, P(n)$$

- where the universe of discourse is the set of positive integers.

Mathematical Induction

- It is a method of mathematical proof typically used to establish that
- a given statement is true of all natural numbers, or
- otherwise is true of all members of an infinite sequence.

Mathematical induction can be analized as the domino effect



1. The first domino will fall.
2. Whenever a domino falls, its next neighbor will also fall.

Then you can conclude that *all* dominos will fall.

Steps of Mathematical Induction

1. **Basis Step**: showing that the statement holds when $n = 0$ or any initial value.

1. **Inductive step**: showing that ***if*** the statement holds for $n = m$, ***then*** the same statement also holds for $n = m + 1$.

- The proposition following the word "if" is called the **induction hypothesis**.

Example: Mathematical Induction

- Suppose we wish to prove the statement:

$$1+2+3+\cdots+n=\frac{n(n+1)}{2}$$

- for all natural numbers *n*.

Example: Mathematical Induction

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

- This is a simple formula for the sum of the natural numbers up to the number *n*.

$$1+2+3+\cdots+n=\frac{n(n+1)}{2}$$

- Check if it is true for *n* = 1.
- The sum of the first 1 natural numbers is 1, and $1=\frac{1(1+1)}{2}$
- So the statement is true for *n* = 1.
- The statement is defined as P(*n*), and P(1) holds.

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

- Now we have to show that if the statement holds when $n = m$, then
- it also holds when $n = m + 1$.
- Assume the statement is true for $n = m$

$$1 + 2 + \cdots + m = \frac{m(m+1)}{2}$$

$$1+2+3+\cdots+n=\frac{n(n+1)}{2}$$

$$1+2+\cdots+m=\frac{m(m+1)}{2}$$

- Under this assumption, it must be shown that P(k+1) is true, namely, that

$$1+2+\cdots+m+(m+1)=\frac{(m+1)((m+1)+1)}{2}=\frac{(m+1)(m+2)}{2}$$

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

- Adding *m* + 1 to both sides gives

$$1 + 2 + \cdots + m + (m+1) = \frac{m(m+1)}{2} + (m+1)$$

$$= \frac{m(m+1)}{2} + \frac{2(m+1)}{2}$$

$$= \frac{(m+2)(m+1)}{2}$$

- This last equation shows that P(m+1) is true.

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

- Symbolic $\rightarrow$ ally, we have shown that:

$$P(m) \Rightarrow P(m+1)$$

- The inductive steps are expressed as the following rule of inference

$$[P(1) \wedge \forall m\ (P(m) \rightarrow P(m+1))] \rightarrow \forall n\ P(n)$$

Some Common Proof Techniques

Direct proof: where the conclusion is established by logically combining the axioms, definitions and earlier theorems.

Proof by induction: where a *base case* is proved, and an *induction rule* used to prove an (often infinite) series of other cases.

Some Common Proof Techniques

Proof by contradiction (also known as *reductio ad absurdum*): where it is shown that if some statement were false, a logical contradiction occurs, hence the statement must be true.

Some Common Proof Techniques

Proof by construction: constructing a concrete example with a property to show that something having that property exists.

Proof by exhaustion: where the conclusion is established by dividing it into a finite number of cases and proving each one separately.

Some Common Proof Techniques

- A *combinatorial proof* establishes the equivalence of different expressions by showing that they count the same object in different ways.
- Usually a one-to-one correspondence is used to show that the two interpretations give the same result.

Some Common Proof Techniques

- A statement which is thought to be true but has not been proven yet is known as a conjecture.
- In most axiom systems, there are statements which can neither be proven nor disproven.

The Principle of Mathematical Induction

- $\mathcal{P}$ is true for some particular integer n_0.
- If $k \geq n_0$ is an integer and $\mathcal{P}$ is true for k, then P is true for the next integer $k + 1$ (Induction hypothesis).
- Then $\mathcal{P}$ is true for all integers $n \geq n_0$.

The Principle of Mathematical Induction

- A proof by mathematical induction that $\mathcal{P}(n)$ is true for every positive integer n consists of two steps:
- Basis step: The proposition $\mathcal{P}(1)$ is shown to be true.
- Inductive step: The implication

$$\mathcal{P}(k) \rightarrow \mathcal{P}(k+1)$$

is shown to be true for every positive integer k.

Example 1 using The Principle of Mathematical Induction

- Prove that for any integer $n \geq 1$ the sum of the odd integers from 1 to $2n - 1$ is n^2.

- The sum in question is often written

$$1 + 3 + 5 + \ldots + (2n - 1).$$

Odd numbers

A formula of the general term

Example 1 using The Principle of Mathematical Induction

(2n – 1) ← Evaluating the general term, we can obtain all numbers of this serie

- n = 1

$$1 = (2(1) - 1) = 2 - 1$$

- n = 2

$$3 = (2(2) - 1) = 4 - 1$$

...

Example 1 using The Principle of Mathematical Induction

$$1 + 3 + 5 + \ldots + (2n - 1) = \sum_{i=1}^{n} (2i - 1)$$

- We can prove that, for all integers $n \geq 1$,

$$1 + 3 + 5 + \ldots + (2n - 1) = n^2$$

$$\sum_{i=1}^{n} (2i - 1) = n^2$$

Example 1 using The Principle of Mathematical Induction

Step 1, $n_0 = 1$

When $n = 1$,

$1 + 3 + 5 + \ldots + (2n - 1)$ means

"the sum of the odd integers from 1 to $2(1) - 1 = 1$."

Example 1 using The Principle of Mathematical Induction

Step 2, Suppose k is an integer, $k \geq 1$, and the statement is true for $n = k$

suppose

$$\underline{1 + 3 + 5 + \ldots + (2k - 1) = k^2}$$

↑ **Induction Hypothesis**

Example 1 using The Principle of Mathematical Induction

Now, show that the statement is true for the next integer, $n = k + 1$

$$1 + 3 + 5 + \ldots + (2(k+1) - 1) = (k + 1)^2$$

If $(2(k+1) - 1) = 2k + 1$, then

$$1 + 3 + 5 + \ldots + (2k+1) = (k + 1)^2$$

Example 1 using The Principle of Mathematical Induction

- The sum on the left is the sum of the odd integers from 1 to 2k + 1.
- This is the sum of the odd integers from 1 to 2k – 1, plus the next odd integer, 2k + 1

$$1 + 3 + 5 + \ldots + (2k+1)$$

$$= [1 + 3 + 5 + \ldots + (2k - 1)] + (2k + 1)$$

Example 1 using The Principle of Mathematical Induction

- By induction hypothesis, we know that

$$1 + 3 + 5 + \ldots + (2k+1)$$

$$= [1 + 3 + 5 + \ldots + (2k - 1)] + (2k + 1)$$

$$= k^2 + (2k + 1) = (k + 1)^2$$

Example 1 using The Principle of Mathematical Induction

- This is the result wanted

$\underline{1 + 3 + 5 + \ldots + (2k - 1)} + (2k + 1) = \underline{k^2} + (2k + 1)$

- Conclusion: By the Principle of Mathematical induction

  $1 + 3 + 5 + \ldots + (2n - 1) = n^2$, <u>is true for all $n \geq 1$</u>

Example 2 using The Principle of Mathematical Induction

- Prove that for any integer $n \geq 1$,

$$1^2 + 2^2 + 3^2 + \dots + n^2$$

$$= (n(n+1)(2n+1))/6$$

Example 2 using The Principle of Mathematical Induction

Solution:

- Step 1, $n = 1$

the sum of the integers from 1^2 to 1^2 is 1^2.

$$(1(1 + 1)(2.1 + 1))/6 = 6/6 = 1$$

- So the statement is true for $n = 1$.

Example 2 using The Principle of Mathematical Induction

- Step 2, suppose $k \geq 1$, and the statement is true for $n = k$,

$$\underline{1^2 + 2^2 + 3^2 + \ldots + k^2 = (k(k+1)(2k+1))/6}$$

↑ Induction Hypothesis

Show that the statement is true for $n = k + 1$

Example 2 using The Principle of Mathematical Induction

for n=k + 1

$$1^2 + 2^2 + 3^2 + \ldots + (k + 1)^2$$

$$= ((k+1)((k+1) + 1)(2(k+1) + 1))/6$$

$$= ((k+1)(k+2)(2k+3))/6$$

Example 2 using The Principle of Mathematical Induction

$$((k+1)((k+1)+1)(2(k+1)+1))/6$$

$$= ((k^2+2k+1+k+1)(2k+3))/6$$

$$= ((k^2+3k+2)(2k+3))/6$$

$$= ((k+2)(k+1)(2k+3))/6$$

Example 2 using The Principle of Mathematical Induction

$1^2 + 2^2 + 3^2 + \ldots + (k + 1)^2$

$= 1^2 + 2^2 + 3^2 + \ldots + k^2) + (k + 1)^2$

$= (k(k+1)(2k+1))/6 \quad + (k + 1)^2$

$= (k(k+1)(2k+1) \quad + 6(k + 1)^2)/6$

Example 2 using The Principle of Mathematical Induction

$1^2 + 2^2 + 3^2 + \ldots + (k + 1)^2$

$= (k(k+1)(2k+1) + 6(k + 1)^2) / 6$

$= (k+1) [k(2k+1) + 6(k + 1)] / 6$

$= (k + 1)[2k^2 + 7k + 6] / 6$

$= ((k + 1)(k + 2)(2k + 3)) / 6$

Example 2 using The Principle of Mathematical Induction

- This is the result wanted

$$1^2 + 2^2 + 3^2 + \ldots + (k + 1)^2$$

$$= ((k + 1)(k + 2)(2k + 3)) / 6$$

- Conclusion: By the Principle of Mathematical induction

$1^2 + 2^2 + 3^2 + \ldots + n^2$
$= (n(n + 1)(2n + 1))/6$, is true for all $n \geq 1$

Example 3 using The Principle of Mathematical Induction

- Prove that for any integer $n \geq 1$,

$2^{2n} - 1$ is divisible by 3.

Example 3 using The Principle of Mathematical Induction

Solution:

- Step 1, n = 1

$$2^{2(1)} - 1 = 2^2 - 1 = 4 - 1 = 3$$

3 is divisible by 3

- So the statement is true for n = 1.

Example 3 using The Principle of Mathematical Induction

- Step 2, suppose $k \geq 1$, and the statement is true for $n = k$,

  $\underline{2^{2k} - 1 \text{ is divisible by } 3}$

  ↑ Induction Hypothesis

Example 3 using The Principle of Mathematical Induction

Show that the statement is true for n= k + 1

$$2^{2(k+1)} - 1 = (2^{2k}.2^{2}) - 1 = 4(2^{2k}) - 1$$

$2^{2k} - 1 = 3t$ for some integer t (by induction hypothesis)

So $2^{\mathbf{2k}} = 3t + 1$

Example 3 using The Principle of Mathematical Induction

$$2^{2(k+1)} - 1 = 4(2^{2k}) - 1$$
$$= 4(3t + 1) - 1$$
$$= 12t + 4 - 1 = 12t + 3$$
$$= 3(4t + 1)$$

Thus, $2^{2(k+1)} - 1$ is divisible by 3.

- Conclusion: By the Principle of Mathematical induction

$2^{2n} - 1$ is divisible by 3 <u>for all $n \geq 1$</u>

Example 4 using The Principle of Mathematical Induction

- Prove that $2^n < n!$ for all $n \geq 4$,

Solution:

- Step 1, $n_0 = 4$

  $2^4 = 16 < 4! = 24$

  Thus, the statement is true for n_0.

Example 4 using The Principle of Mathematical Induction

- Step 2, suppose $k \geq 4$, and the statement is true for $n = k$,

  $\underline{2^k < k!}$

  ↑ Induction Hypothesis

Show that the statement is true for $n=k+1$

Example 4 using The Principle of Mathematical Induction

n = k + 1, prove that $2^{k+1} < (k + 1)!$

Multiplying both sides of the inequality $2^k < k!$ by 2

$$2 \cdot 2^k < 2 \cdot k!$$
$$< (k + 1) \cdot k! = (k + 1)!$$

P(k+1) is true when p(k) is true, so $2^n < n! \ \forall \ n \geq 4$

The Principle of Mathematical Induction (Strong Form)

- $\mathcal{P}$ is true for some integer n_0;
- if $k \geq n_0$ is any integer and $\mathcal{P}$ is true for all integers ℓ in the range $n_0 \leq \ell < k$, then it is true also for k.
- Then $\mathcal{P}$ is true for all integers $n \geq n_0$.

The Principle of Mathematical Induction (Strong Form)

- $\mathcal{P}(n)$ is true for all positive integers n:
- **Basis Step**: The proposition $\mathcal{P}(1)$ is shown to be true.
- **Inductive Step**: It is shown that

$$[\ \mathcal{P}(1) \wedge \mathcal{P}(2) \wedge \ldots \wedge \mathcal{P}(k)\ ] \rightarrow \mathcal{P}(k + 1)$$

is true for every positive integer k.

The Principle of Mathematical Induction

(Strong Form)	(Weak Form)
■ Assume the truth of the statement for **all** integers less than some integer, and ■ prove that the statement is true for that integer.	■ Assumed the truth of the statement for just **one** particular integer, and ■ prove it true for the next largest integer.

Example 5 using The Principle of Mathematical Induction (Strong Form)

- Prove that every natural number $n \geq 2$ is either prime or the product of prime numbers.

Example 5 using The Principle of Mathematical Induction (Strong Form)

Solution:

- **Basis Step:** $n_0 = 2$, the assertion of the theorem is true.

- Suppose that every integer ℓ in the interval $2 \leq \ell < k$ is either prime or the product of primes.

Example 5 using The Principle of Mathematical Induction (Strong Form)

- **Inductive Step:**

- If k is prime, the theorem is proved.
- if k is not prime, then k can be factored $k = ab$, where a and b are inegers satisfing $2 \leq a, b < k$.

- By induction hypothesis, each of a and b is either prime or the product of primes.
- k is the product of primes, as required.

Example 5 using The Principle of Mathematical Induction (Strong Form)

- Conclusion: By the Principle of Mathematical Induction,

  we conclude that every $n \geq 2$ is prime or the product of two primes.

Example 5 using The Principle of Mathematical Induction (Strong Form)

- An store sells envelopes in packages of **five** and **twelve** and want to by **n** envelopes.

- Prove that for every **n** $\geq$ **44** the store can sell you exactly **n** envelopes

  (assuming an unlimited supply of each type of envelope package).

Example 5 using The Principle of Mathematical Induction (Strong Form)

- Solution:

Given that envelopes are available in packages of 5 and 12, we wish to show an order for n envelopes can be filled exactly, provided $n \geq 44$.

Example 5 using The Principle of Mathematical Induction (Strong Form)

- Assume that $k > 44$ and that an order for ℓ envelopes can be filled if $44 \leq \ell < k$

  Our argument will be that $k = (k - 5) + 5$

- By the induction hypothesis, $k - 5$ envelopes can be purchased with packages of five and twelve so, by adding one more package of five, we can purchase k.

Example 5 using The Principle of Mathematical Induction (Strong Form)

- We can apply the induction hypothesis if

$\ell = k - 5$

$k - 5 \geq 44$

$k \geq 44 + 5 = 49$

$k \geq 49$

- The remaining cases, $k = 45, 46, 47, 48$ are checked individually(Note: $44 \leq \ell < k$).

Example 5 using The Principle of Mathematical Induction (Strong Form)

- 45 = 9 packages of five envelopes
- 46 = 3 three packages of twelve and
  2 package of five.
- 47 = 1 package of twelve and
  7 packages of five.
- 48 = 4 packages of twelve.

Mathematical Induction and Well Ordering

- The Well ordering principle states that "any nonempty set of natural numbers has a smallest element."

- A set containing just one element has a smallest member, the element itself, so the Well-Ordering Principle is true for sets of size $n_0 = 1$.

Mathematical Induction and Well Ordering

- Suppose this principle is true for sets of size k. Assume that <u>any set of k natural numbers has a smallest member.</u>

- Given a set S of k + 1 numbers, remove one element a. The remaining k numbers have a smallest element, say b, and the smaller of a and b is the smallest element of S.

Mathematical Induction and Well Ordering

- We may use the Well-Ordering Principle to prove the Principle of Mathematical Induction (weak form).

Mathematical Induction and Well Ordering

- Suppose that $\mathcal{P}$ is a statement involving the integer n that we wish to establish for all integers greater than or equal to some given integer n_0. Assume:

1. $\mathcal{P}$ is true for $n = n_0$, and
2. If k is an integer, $k \geq n_0$, and $\mathcal{P}$ is true for k, then P is also true for $k + 1$.

How the Well-Ordering Principle show that $\mathcal{P}$ is true for all $n \geq n_0$?

1. Assume $n_0 \geq 1$.

2. If $\mathcal{P}$ is not true for all $n \geq n_0$, then the set S of natural numbers $n \geq n_0$, for which $\mathcal{P}$ is false is not empty.

3. By the well_ordering Principle, S has a smallest element a. Now $a \neq n_0$ because was established that P is true for $n = n_0$.

How the Well-Ordering Principle show that $\mathcal{P}$ is true for all $n \geq n_0$?

1. Thus $a > n_0$, $a - 1 \geq n_0$.
2. Also, $a - 1 < a$. By minimality of a, $\mathcal{P}$ is true for $k = a - 1$.
3. We are foced to conclude that our starting assumption is false: $\mathcal{P}$ must be true for all $n \geq n_0$.

How the Well-Ordering Principle show that $\mathcal{P}$ is true for all $n \geq n_0$?

3. By assumption 2, $\mathcal{P}$ is true for k + 1 = a, a contradiction.

If k is an integer, $k \geq n_0$, and $\mathcal{P}$ is true for k, then P is also true for k + 1.

We are foced to conclude that our starting assumption is false: $\mathcal{P}$ must be true for all $n \geq n_0$.

Mathematical Induction and Well Ordering

- The priciples of Well-Ordering and Mathematical Induction (weak form) are equivalent.

Recursively Defined Sequences

- Suppose n is a natural number. How should define 2^n?

$$2^n = \underbrace{2 \cdot 2 \cdot 2 \; \ldots \; 2}_{n\ 2\text{'s}}$$

$2^1 = 2$, and for $k \geq 1$, $2^{k+1} = 2 \cdot 2^k$

a recursive definition ↑

Recursively Defined Sequences

- n! is a recursive sequence

$0! = 1$ and

for $k \geq 0$, $(k + 1)! = (k+1)k!$

Recursively Defined Sequences

- A sequence is a function whose domain is some infinite set of integers (often N) and whose range is a set of real number(R).

Example: The sequence that is the function

$$f: N \to R \text{ defined by } f(n) = n^2$$

is described by the list 1, 4, 9, 16, ...

Recursively Defined Sequences

1, 4, 9, 16, ...

- The numbers in the list (the range of the function) are called the terms of the sequence.

- The sequence 2, 4, 8, 16, ... can be defined recursively like this

  $a_1 = 2$ and for $k \geq 1$, $a_{k+1} = 2a_k$

Recursively Defined Sequences

- The equation $a_{k+1} = 2a_k$ defines one member of the sequence in terms of the previus.
- It is called a recurrence relation.
- $a_1 = 2$ is called an initial condition.
- $a_2 = 2a_1 = 2(2) = 4.$ $k = 2$
- $a_3 = 2a_2 = 2(4) = 8.$ $k = 3$

Recursively Defined Sequences

- There are other posible recursive definitions that describe the same sequence
- $a_0 = 2$ and for $k \geq 0$, $a_{k+1} = 2a_k$

or

- $a_1 = 2$ and for $k \geq 0$, $a_k = 2a_{k-1}$

Example1: Recursively Defined Sequences

- Write down the first six terms of the sequence defined by

  $a_1 = 1$, $a_{k+1} = 3a_k + 1$ for $k \geq 1$. Guess a formula for a_n, and prove that your formula is correct.

Example1: Recursively Defined Sequences

- Solution

$a_1 = 1$,

$a_2 = 3a_1 + 1 = 3(1) + 1 = 4$

$a_3 = 3a_2 + 1 = 3(4) + 1 = 13$

$a_4 = 40$

$a_5 = 121$

$a_6 = 364$

Example 2: Recursive Function

- Find the formula for a_n, given $a_1 = 1$ and $a_{k+1} = 3a_k + 1$ for $k \geq 1$, without guesswork.
- Hint: Use the formula:

$$a_{k+1} = \frac{1}{2} 3^{k+1} - 3/2 + 1 = \frac{1}{2}(3^{k+1} - 1)$$

Example 2: Recursive Function

- Since $a_n = 3a_{n-1} + 1$ and $a_{n-1} = 3a_{n-2} + 1$,

  $$a_n = 3a_{n-1} + 1 = 3(3a_{n-2} + 1) + 1$$
  $$= 3^2a_{n-2} + (1 + 3 + 3^2).$$

  ↑ ↑

- First part has the form 3^ka_{n-k}
- Second part is the sum of geometric series

$$a_n = 3^{n-1}a_1 + (1 + 3 + 3^2 + \ldots + 3^{n-2}).$$

$a_1 = 1$ and

$$1 + 3 + 3^2 + \ldots + 3^{n-2} = 1(1 - 3^{n-1})/(1 - 3)$$

$$= ½\ (3^{n-1} - 1)$$

$$a_n = 3^{n-1} + ½\ (3^{n-1} - 1)$$

$$= ½\ (2\ .\ 3^{n-1} + 3^{n-1} - 1)$$

$$= ½\ (3.\ 3^{n-1} - 1)$$

$$\boxed{= ½\ (3^n - 1)}$$

Example 3 : Recursive Functions

- Give an inductive definition of the factorial function $f(n) = n!$

- The factorial function can be defined by specifying the initial value of this function, $f(0) = 1$, and giving a rule for finding $f(n+1)$ from $f(n)$.

Example 3: Recursive Functions

- $f(n+1) = (n+1) . f(n)$

↑ Rule to determine a value of the factorial function

$f(n+1) = (n+1).\ f(n)$

- $f(5) = 5!$

  $f(5) = 5 \cdot f(4)$

  $= 5 \cdot 4 \cdot f(3)$

  $= 5 \cdot 4 \cdot 3 \cdot f(2)$

  $= 5 \cdot 4 \cdot 3 \cdot 2 \cdot f(1)$

  $= 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot f(0)$

  $= 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 1$

  $= 120$

Example 4: Recursive Functions

- Give a recursive definition of a^n, with

  $a \neq 0 \mid a \in R \quad n \geq 0 \mid n \in Z^+$.

- The recursive definition contains two parts

First: $a^0 = 1$

Second: $a^{n+1} = a \cdot a^n$, for $n=0, 1, 2, \ldots, n$

These two equations uniquely define a^n for all nonnegative integers n.

Example 5: Recursive Functions

- Give a recursive definition of

$$\sum_{k=0}^{n} a^k$$

Example 5: Recursive Functions

- The recursive definition contains two parts

The First part:

$$\sum_{k=0}^{n} a_k = a_0$$

The Second part:

$$\sum_{k=0}^{n+1} a_k = \left[\sum_{k=0}^{n} a_k \right] + a_{n+1}$$

Example 6: Recursive Functions

- Find Fibonacci numbers, f_0, f_1, f_2, ..., are defined by the equations $f_0=0$, $f_1=1$, and

$$f_n = f_{n-1} + f_{n-2} \qquad \text{for } n = 2, 3, 4, \ldots$$

- Find Fibonacci numbers f_2, f_3, f_4, f_5, and f_6

Example 6: Recursive Functions

- Find Fibonacci numbers f_2, f_3, f_4, and f_5

$f_n = f_{n-1} + f_{n-2}$ $f_0 = 0$, $f_1 = 1$,

$f_2 = f_1 + f_0 = 1 + 0 = 1$

$f_3 = f_2 + f_1 = 1 + 1 = 2$

$f_4 = f_3 + f_2 = 2 + 1 = 3$

$f_5 = f_4 + f_3 = 3 + 2 = 5$

The Characteristic Polynomial

The homogeneous recurrence relation

$a_n = ra_{n-1} + sa_{n-2}$ can be rewritten in the form

$$a_n - ra_{n-1} - sa_{n-2} = 0,$$

Which can be associated with $\mathbf{x^2 - rx - s}$

- This polynomial is called the characteristic polynomial of the recurrence relation

The Characteristic Polynomial

$$x^2 - rx - s$$

- Its roots are called the characteristic polynomial roots of the recurrence relation.

Example: The Characteristic Polynomial

- The recurrence relation $a_n = 5a_{n-1} - 6a_{n-2}$ has the characteristic polynomial

$$a_2 - 5a_{2-1} - 6a_{2-2} = 0,$$

$$(x - 2)(x - 3)$$

$$x^2 - 5x + 6$$

and characteristic roots 2 and 3.

Theorem the Characteristic Polynomial

- Let x_1 and x_2 be the roots of the polynomial $x^2 - rx - s$. Then the solution of the recurrence relation $a_n = ra_{n-1} + sa_{n-2}$, $n \geq 2$ is

$$a_n = \begin{cases} c_1 x_1^n + c_2 x_2^n & \text{if } x_1 \neq x_2 \\ c_1 x^n + c_2 n x^n & \text{if } x_1 = x_2 = x \end{cases}$$

where c_1 and c_2 are constants defined by initial conditions

Example: Theorem the Characteristic Polynomial

- Solve the recurrence relation

$a_n = 5a_{n-1} - 6a_{n-2}$, $n \geq 2$ given $a_0 = -3$, $a_1 = -2$.

The characteristic polynomial $x^2 - 5x + 6$.
has the roots $x_1 = 2$, $x_2 = 3$ $(x_1 \neq x_2)$

$$a_n = c_1(x_1^n) + c_2(x_2^n)$$
$$a_n = c_1(2^n) + c_2(3^n)$$
$$a_0 = -3 = c_1(2^0) + c_2(3^0)$$
$$a_1 = -2 = c_1(2^1) + c_2(3^1)$$

Example: Theorem the Characteristic Polynomial

$a_n = 5a_{n-1} - 6a_{n-2}$, $n \geq 2$ and $a_0 = -3$, $a_1 = -2$.

Solve the following system of equations

$$c_1 + c_2 = -3$$

$$2c_1 + 3c_2 = -2$$

$c_1 = -7$, $c_2 = 4$, so the solution is

$$a_n = -7(2^n) + 4(3^n)$$

Arithmetic Sequences

- The arithmetic sequence with first term a and common difference d is the sequence defined by

$$a_1 = a \quad \text{and, for } k \geq 1, \quad a_{k+1} = a_k + d$$

and takes the form

$$a,\ a + d,\ a + 2d,\ a + 3d,\ \ldots$$

Arithmetic Sequences

- For $n \geq 1$, the nth term of the sequence is

$$a_n = a + (n - 1)d$$

- The sum of n terms of the arithmetic sequence with first term a and common difference d is

$$S = n / 2\ [2a + (n - 1)d]$$

Arithmetic Sequences

- The first 100 terms of the arithmetic sequence -17, -12, -7, 2, 3, ... have the sum

$$S = n/2\ [2a + (n - 1)d]$$

$$S = 100/2\ [2(-17) + (100 - 1)5]$$

$$S = 50\ [-34 + (99)5]$$

$$S = \underline{23{,}050}$$

Arithmetic Sequences

- The 100th term of this sequence is

$$a_n = a + (n - 1)d$$

$$a_{100} = a + (n - 1)d$$

$$a_{100} = -17 + (100 - 1)5$$

$$a_{100} = -17 + (99)5$$

$$a_{100} = -17 + 495 = \underline{478}$$

Geometric Sequences

- The geometric sequence with first term a and common ratio r is the sequence defined by

$$a_1 = a \quad \text{and, for } k \geq 1, \quad a_{k+1} = r.a_k$$

and takes the form

$$a,\ ar,\ ar^2,\ ar^3,\ ar^4,\ \ldots$$

Geometric Sequences

- The n^{th} term being

$$a_n = a \cdot r^{n-1}$$

- The sum S of n terms of the geometric sequence, provided $r \neq 1$ is

$$S = a(1 - r^n) / (1 - r)$$

Geometric Sequences

- The sum of 29 terms of the geometric sequence with a = 8^{12} and r = -1/2 is

$$S = a(1 - r^n) / (1 - r)$$

$$S = 8^{12}(1 - (-½)^{29}) / (1 - (-½))$$

$$S = (2^{36}(1 + (½)^{29}) / 3/2$$

$$S = (2^{36} + 2^7) / 3/2 = \underline{1/3\ (2^{37} + 2^8)}$$

$$S = 45812984576$$

Recurrence Relations

- There is procedure for solving recurrence relations of the form

$$a_n = ra_{n-1} + sa_{n-2} + f(n)$$

where r and s are constants and f(n) is some function of n.

Recurrence Relations

$$a_n = ra_{n-1} + sa_{n-2} + f(n)$$

- Such recurrence relation is called a <u>second-order linear recurrence relation with constant coefficients.</u>

if $f(n) = 0$, the relation is called homogeneous.

Second-Order Linear Recurrence Relation with Constant Coefficients

$$a_n = ra_{n-1} + sa_{n-2} + f(n)$$

- Second-order: a_n is defined as a function of the two terms preceding it.
- Linear: the terms a_{n-1} and a_{n-2} appear by themselves, to the first power, and with constant coefficient.

Examples: Second-order linear recurrence relation with constant coefficients

$$a_n = ra_{n-1} + sa_{n-2} + f(n)$$

1. The Fibonacci sequence:

   $a_n = a_{n-1} + a_{n-2}$, $r = s = 1$

2. $a_n = 5a_{n-1} + 6a_{n-2} + n$,

   $r = 5$, $s = 6$, $f(n) = n$.

3. $a_n = 3a_{n-1}$.

   Homogeneous with $r = 3$, $s = 0$

Topics covered

- Mathematical Induction
- Recursively Defined Sequences.
- Solving Recurrence Relations.

Reference

- “Discrete Mathematics with Graph Theory”, Third Edition, E. Goodaire and Michael Parmenter, Pearson Prentice Hall, 2006. pp 147-183.