Foundations of Discrete Mathematics

Chapters 6

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Principle of Counting

- This chapter will illustrate some basic principles of counting.

- The number of elements of a set S (|S|) in various combinations (union, intersection, and difference) of finite sets will be considered.

Principle of Counting

- Let A and B be subsets of a finite universal set U. Then

a) $|A \cup B| = |A| + |B| - |A \cap B|$

b) $|A \cap B| \leq \min\{|A|, |B|\}$, the minimum of $|A|$ and $|B|$.

c) $|A \setminus B| = |A| - |A \cap B| \geq |A| - |B|$

Principle of Counting

d) $|A^c| = |U| - |A|$

e) $|A \oplus B| = |A \cup B| - |A \cap B|$

$= |A| + |B| - 2|A \cap B|$

$= |A \setminus B| + |B \setminus A|$

f) $|A \times B| = |A| \times |B|$

Proof of Principle of Counting

a) $|A \cup B| = |A| + |B| - |A \cap B|$

If $A = \varnothing$, the $A \cap B = \varnothing$, and $|A| = |A \cap B| = 0$

So the results holds

$$|\varnothing \cup B| = |\varnothing| + |B| - |\varnothing \cap B|$$

$$|B| = |\varnothing| + |B| - 0$$

$$|B| = |B|$$

Proof of Principle of Counting

a) $|A \cup B| = |A| + |B| - |A \cap B|$

Suppose $A \cap B = \emptyset$, Let $A = \{a_1, a_2, ..., a_r\}$ and $B = \{b_1, b_2, ..., b_s\}$. Then

$$| A \cup B| = \{a_1, a_2, ..., a_r, b_1, b_2, ..., b_s\}$$

Since there is no repetition among the elements listed

$$| A \cup B| = r + s = |A| + |B|$$

$$= |A| + |B| - | A \cap B|$$

Principle of Counting

$$|A \cup B \cup C| = |A \cup (B \cup C)|$$

$$= |A| + |B \cup C| - |A \cap (B \cup C)|$$

$$= |A| + |B \cup C| - |(A \cap B) \cup (A \cap C)|$$

$$= |A| + [|B| + |C| - |B \cap C|]$$
$$- [|(A \cap B)| + |A \cap C| - |(A \cap B) \cap (A \cap C)|]$$

$$= |A| + |B| + |C| - |B \cap C| - |A \cap B| - |A \cap C| + |A \cap B \cap C|$$

Principle of Counting

Important properties

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$(A \cap B) \cap (A \cap C) = A \cap B \cap C$$

Example 1: Principle of Counting

- Glenys is thinking about registering of a course in data analysis.

- Of the 100 people have been registered, 80 have own personal digital assistant (PDAs) and three-quarters of the group are men.

Example 1: Principle of Counting

a) Estimates the number of women who do not have PADS. How large might this number b? How small?

b) How many of the men registered in this course could conceivably own PADS?

Example 1: Principle of Counting

Introduce the following sets:

- U set of all registrants, $|U| = 100$
- M set of all male registrants, $|M| = 75$
- P set of those registrants who own a PAD, $|P| = 80$

- The set of women who do not have a PAD is

  $M^c \cap P^c$

Example 1: Principle of Counting

- The size of the following sets is given

  $|U| = 100$, $|M| = 75$, and $|P| = 80$

- The set of women who do not have a PAD is $M^c \cap P^c$

Example 1: Principle of Counting

a) The number of women without PAD

$M^c \cap P^c = (M \cup P)^c$ ← one of the laws of De Morgan.

$|M^c \cap P^c| = |(M \cup P)^c| = |U| - |M \cup P|$

$|M^c \cap P^c| = |(M \cup P)^c| = 100 - |M \cup P|$

Example 1: Principle of Counting

a) How big is $M \cup P$ (cont.)?

$$|M \cup P| = |M| + |P| - |M \cap P|$$
$$= 75 + 80 - |M \cap P|$$
$$= 155 - |M \cap P|$$

Therefore,

$$|M^c \cap P^c| = 100 - |M \cup P|$$
$$= 100 - (155 - |M \cap P|)$$
$$= |M \cap P| - 55$$ ← women without PADs

Example 1: Principle of Counting

a) How big is $M \cup P$ (cont.)?

$|M \cap P| \leq |M| = 75$, so

$$|M^c \cap P^c| \leq 75 - 55 = 20.$$

conceivably,

$|M^c \cap P^c| = |M \cap P| - 55$

$$|M^c \cap P^c| = 0 \text{ if } |M \cap P| = 55$$

So the number of women without PADs is

$$0 \leq |M^c \cap P^c| \leq 20$$

Example 1: Principle of Counting

b) How many of the men could conceivably own PADs?

$|M \cap P| - 55 \geq 0$ ← women without PADs

$|M \cap P| \geq 55$ ← at least 55 men must own PADS

Example 1: Principle of Counting

b) (cont.)

The upper bound for $|M \cap P|$ is 75

Suppose that $|M \cap P| = 72$ then

$|M \cup P| = |M| + |P| - |M \cap P| = 75 + 80 - 72$

$= \underline{83}$

There are 83 people in the class who are either men or own a PDA

Example 1: Principle of Counting

c) How many men registered in the class would not have PADs?

$$|M \setminus P| = |M| - |M \cap P| = 75 - 72 = 3$$

Example: Principle of Counting

d) How many of the owner of PADs are women?

$$|P \setminus M| = |P| - |P \cap M| = 80 - 72 = 8$$

Example 1: Principle of Counting

e) How many of the those registered are either men without PADs or women with PADS?

$$|M \oplus P| = |P \setminus M| + |P \setminus M| = 3 + 8 = 11$$

Example 1: Principle of Counting

f) How many of people are either men or owners of a PAD?

$|M \oplus P| = |M \cup P| - |M \cap P| = 83 - 72 = 11$

or

$$|M \oplus P| = |M| + |P| - 2|M \cap P|$$
$$= 75 + 80 - 2(72)$$
$$= 155 - 144 = 11$$

Principle of Inclusion -Exclusion

Given a finite number of finite sets A_1,. A_2, ..., A_n.
The number of elements in the union $A_1 \cup A_2 \cup \ldots \cup A_n$ is $| A_1 \cup A_2 \cup \ldots \cup A_n |$

$$= \sum |A_i| - \sum |A_i \cap A_j| + \sum |A_i \cap A_j \cap A_k| - \ldots + (-1)^{n+1} | A_1 \cap A_2 \cap \ldots \cap A_n|$$

Where the first sum is over all i, the second sum is over all pairs i, j with i < j, the third sum is over all triples i, j, k with i<j<k, and so forth.

Example: Principle of Inclusion-Exclusion

Of 30 PCs owned by faculty members is a certain university department, 20 do not have A drives, 8 have 19-inch monitors, 25 are running Windows XP, 20 have at least two of these properties, and six have all three.

a) How many PCs have at least one of these properties?
b) How many have none of these properties?
c) How many have exactly one property?

Example: Principle of Inclusion-Exclusion

Solution:

- U set of PC owned by faculty members, $|U| = 30$
- A set of PCs without A drives, $|A| = 20$
- M set of PCs with 19-inch monitors, $|M| = 8$
- X set of PCs running under Windows XP, $|X| = 25$
- $|(A \cap M) \cup (A \cap X) \cup (M \cap X)| = 20$
- $|A \cap M \cap X| = 6$

Example: Principle of Inclusion-Exclusion

Using the Principle of Inclusion-Exclusion

$$
\begin{aligned}
20 &= |(A \cap M) \cup (A \cap X) \cup (M \cap X)| \\
&= |A \cap M| + |A \cap X| + |M \cap X| \\
&\quad - |(A \cap M) \cap (A \cap X)| - |(A \cap M) \cap (M \cap X)| \\
&\quad - |(A \cap X) \cap (M \cap X)| \\
&\quad + |(A \cap M) \cap (A \cap X) \cap (M \cap X)|
\end{aligned}
$$

Each of the last four terms here is $|A \cap M \cap X|$

Example: Principle of Inclusion-Exclusion

$20 = |(A \cap M) + (A \cap X) + (M \cap X)| - 2|A \cap M \cap X|;$

therefore

$$= |A \cap M| + |A \cap X| + |M \cap X|$$

$$= 20 + 2(6) = 32$$

Example: Principle of Inclusion-Exclusion

a) The number of PCs with at least one property is

$|A \cup M \cup X|$

$= |A| + |M| + |X|$

$- |A \cap M| - |A \cap X| - |M \cap X| + |A \cap M \cap X|$

$= 20 + 8 + 25 - |(A \cap M) + (A \cap X) + (M \cap X)| + 6$

$= 59 – 32 = 27$

Example :Principle of Inclusion-Exclusion

b) How many PCs have none of these properties?

30 – 27 = 3 PCs have none of the specified properties.

c) How many PCs have exactly one property?

27 the number of PC with at least one property
20 the number of PC with at least two properties

27 – 20 = 7.

Example 2:Principle of Inclusion-Exclusion

Suppose 18 of the 30 personal computers of the previous problem have Pentium III processors, including 10 of those running Windows, all of those with 19-inch monitors, and 15 of those with CD-ROM drives.

Suppose also that every computer has at least one of the four features now specified.

How many have at least three features?

Example 2:Principle of Inclusion-Exclusion

Solution:

- P set of PCs with Pentium III processors, $|P| = 18$
- W set of PCs running with Windows, $|W| = 20$
- M set of PCs with 19-inch monitors, $|M| = 8$
- C set of PCs with CD-ROM, $|C| = 25$

Example 2:Principle of Inclusion-Exclusion

$$n = |(W \cap M \cap C) \cup (W \cap M \cap P) \cup (W \cap C \cap P) \cup (M \cap C \cap P)|$$

$$= |W \cap M \cap C| + |W \cap M \cap P| + |W \cap C \cap P| + |M \cap C \cap P| - 6|(W \cap M \cap C \cap P| + 4|(W \cap M \cap C \cap P| - |W \cap M \cap C \cap P|$$

$$= |W \cap M \cap C| + |W \cap M \cap P| + |M \cap C \cap P| - 3|W \cap M \cap C \cap P|$$

Example 2: Principle of Inclusion-Exclusion

$$30 = |W \cup M \cup C \cup P|$$

$$= |W| + |M| + |C| + |P|$$

$$- |W \cap M| - |W \cap C| - |W \cap P|$$

$$- |M \cap C| - |M \cap P| - |C \cap P|$$

$$+ |W \cap M \cap C| + |W \cap M \cap P|$$

$$+ |W \cap C \cap P| + |M \cap C \cap P| - |W \cap M \cap C \cap P|$$

$$= |W| + |M| + |C| + |P| - |W \cap M| - |W \cap C| - |W \cap P|$$

$$+ |M \cap C| - |M \cap P| - |C \cap P| + n$$

$$+ 2|W \cap M \cap C \cap P|$$

Example 2:Principle of Inclusion-Exclusion

because

$$|W \cap M \cap C| + |W \cap M \cap P|$$
$$+ |W \cap C \cap P| + |M \cap C \cap P|$$
$$= n + 3|W \cap M \cap C \cap P|$$

From example 1

$|W \cap C| + |W \cap M| + |W \cap C| = 32$ and $|W \cap M \cap C| = 6$

Since $M \subseteq P$, we have $W \cap M \cap C \cap P = W \cap M \cap C$, and so $|W \cap M \cap C \cap P| = 6$

Example 2:Principle of Inclusion-Exclusion

$|W \cap M \cap C \cap P| = 6$, Therefore,

$30 = 20 + 8 + 25 + 18 – 32 – 10 – 8 – 15 + n + 2(6)$

So $n = 30 – 71 + 65 -12 = 12$

Example 3: Principle of Inclusion-Exclusion

How many integers between 1 and 300 (inclusive) are

a) Divisible by at least one of 3, 5, 7?

b) Divisible by 3 and by 5 but not by 7?

c) Divisible by 5 but by neither 3 nor 7?

d) Relatively prime to 105?

Example 3:Principle of Inclusion-Exclusion

a) Divisible by at least one of 3, 5, 7?

Solution:

- $A = \{n \mid 1 \le n \le 300,\ 3 \mid n\}$
- $B = \{n \mid 1 \le n \le 300,\ 5 \mid n\}$
- $C = \{n \mid 1 \le n \le 300,\ 7 \mid n\}$

Example 3:Principle of Inclusion-Exclusion

a) Divisible by at least one of 3, 5, 7? (cont.)

For natural numbers a and b, the number of positive integers less than or equal to a and divisible by b is $\lfloor a/b \rfloor$

$|A| = \lfloor 300/3 \rfloor = 100$

$|B| = \lfloor 300/5 \rfloor = 60$

$|C| = \lfloor 300/7 \rfloor = 42$

Example 3:Principle of Inclusion-Exclusion

a) Divisible by at least one of 3, 5, 7? (cont.)

Find $A \cap B$, the set of integers between 1 and 300 that are divisible by both 3 and 5.

3 and 5 are relatively prime numbers, any number divisible by each of them must be divisible by their product.

$$|A \cap B| = \lfloor 300/15 \rfloor = 20$$

Example 3: Principle of Inclusion-Exclusion

a) Divisible by at least one, 3, 5, 7? (cont.)

Find $A \cap B$, $B \cap C$, $A \cap C$, and $A \cap B \cap C$

$|A| = \lfloor 300/3 \rfloor = 100$

$|B| = \lfloor 300/5 \rfloor = 60$

$|C| = \lfloor 300/7 \rfloor = 42$

$|A \cap B| = \lfloor 300/15 \rfloor = 20$

$|B \cap C| = \lfloor 300/35 \rfloor = 8$

$|A \cap C| = \lfloor 300/21 \rfloor = 14$

$|A \cap B \cap C| = \lfloor 300/105 \rfloor = 2$

$|A \cup B \cup C| = 100 + 60 + 42 - 20 - 14 - 8 + 2 = 162$

Example 3:Principle of Inclusion-Exclusion

b) Divisible by 3 and by 5 but not by 7?

Those numbers in $(A \cap B) \setminus C$

A set of cardinality $|A \cap B| - |A \cap B \cap C|$

$= 20 - 2 = 18$

Example 3:Principle of Inclusion-Exclusion

c) Divisible by 5 but by neither 3 nor 7?

Those numbers in $B \setminus (A \cup C)$

A set of cardinality $|B| - |B \cap (A \cup C)$

Since $B \cap (A \cup C) = (B \cap A) \cup B \cap C)$

Example 3: Principle of Inclusion-Exclusion

c) Divisible by 5 but by neither 3 nor 7? (cont.)

The Principle of Inclusion-Exclusion gives

$|B \cap (A \cup C)|$

$= |B \cap A| + |B \cap C| - |(B \cap A) \cap (B \cap C)|$

$= |B \cap A| + |B \cap C| - |B \cap A \cap C|$

Because $(B \cap A) \cap (B \cap C) = B \cap A \cap C$.

Therefore $|B \cap (A \cup C)| = 20 + 8 – 2 = 26$

$|B| - |B \cap (A \cup C)| = |B| - 26 = 60 – 26 = 34$

Example 3:Principle of Inclusion-Exclusion

d) Relatively prime to 105?

- 105 = 3(5)(7), an integer is relatively prime to 105 if and only if it is not divisible by 3, by 5, or by 7.

- 162 integers n in the range $1 \leq n \leq 300$ are divisible by at least one of these numbers,

- 300 – 162 = 138 are not.

The Addition Rule

The number of ways in which precisely one of a collection of mutually exclusive events can occur is the sum of the numbers of ways each event can occur.

Example: Addition Rule

In how many ways can you get a total of six when rolling two dice?

Solution:

The event “get a six” is the union of the mutually exclusive subevents.

A_1: “two 3’s”

A_2: “a 2 and a 4”

A_3: “a 1 and a 5”

Example: Addition Rule

Event A_1 can occur in one way.

A_2 can occur in two ways (depending on which die lands 4)

A_3 can occur in two ways.

So the number of ways to get a six is

$$1 + 2 + 2 = 5$$

Multiplication Rule

The number of ways in which a sequence of events can occur is the product of the numbers of ways in which each individual event can occur.

Example: Multiplication Rule

License plates in the Canadian province of Ontario consist of four letters followed by three of the digits 0-9 (not necessarily distinct).

How many different licenses plates can be made in Orlando?

Example: Multiplication Rule

Solution:

By the multiplication rule, the number of ways in which the four letters can be chosen is $26 \times 26 \times 26 \times 26 = 26^4$, (26 letters in the alphabet).

By the multiplication rule, the number of ways in which the three digits can be chosen is $10 \times 10 \times 10 = 10^3$, (10 digits).

Example: Multiplication Rule

Solution:

The number of different license plates that can be made in Ontario are

$$26^4 \times 10^3 = 467,976,000$$

The Pigeonhole Principle

If n objects are put into m boxes and n > m, then at least one box contains two or more of the objects.

Example 1: The Pigeonhole Principle

Among any group of 367 people, there must be at least two with the same birthday, because there are only 366 possible birthdays.

Example 2: The Pigeonhole Principle

In any group of 27 English words, there must be at least two that begin with the same letter, since there are 26 letter in the English alphabet.

Example 3: The Pigeonhole Principle

How many students must be in a class to guarantee that at lest two students receive the same score on the final exam, if the exam is graded on a scale from 0 to 100 points?

Example 3: The Pigeonhole Principle

Solution:

There are 101 (0- 100) possible scores on the final.

The pigeonhole principle shows that among any 102 students there must be at least 2 students with the same score.

The Pigeonhole Principle (Strong Form)

If n objects are put into m boxes and $n > m$, then
some boxes must contain at least $\lceil a/b \rceil$ objects.

The Pigeonhole Principle (Strong Form)

- To prove the strong form of the Pigeonhole Principle, we establish the truth of its contrapositive.

Note that

$\lceil n/m \rceil < n/m + 1$ and hence $\lceil n/m \rceil - 1 < n/m$

Because, for any real number x,

$$x \leq \lceil x \rceil < x + 1$$

The Pigeonhole Principle (Strong Form)

- Thus, if a box contains fewer than $\lceil n/m \rceil$ objects, then it contains at most $\lceil n/m \rceil - 1$ and so fewer than n/m objects.

- If all m boxes are like this, we account for fewer than

$$m \times n/m = n \text{ objects.}$$

Example 1: The Pigeonhole Principle

- Among 100 people there are at least $\lceil 100/12 \rceil = 9$ who were born in the same moth.

Example 2: The Pigeonhole Principle

- What is the minimum number of students N required in a discrete mathematics class to be sure that at least six will receive the same grade, if there are five possible grades A, B, C, D, and F?

Example 2: The Pigeonhole Principle

Solution:

- $\lceil N/5 \rceil = 6$ ← ceiling
- $N = 5 . 5 + 1 = 26$
- 26 students is the minimum number of students needed to ensure that at least six students will receive the same grade.

Example 3: The Pigeonhole Principle

How many cards must be selected from a standard deck of 52 cards to guarantee that at least three cards of the same suit are chosen?

Solution:

Suppose 4 boxes, and as cards are selected they are placed in the box reserved for cards of that suit.

Example 3: The Pigeonhole Principle

If N cards are selected, there is at least one box containing $\lceil N/4 \rceil$ cards.

At least three cards of one suit are selected if $\lceil N/4 \rceil \geq 3$.

The smallest integer N such that $\lceil N/4 \rceil \geq 3$ is

$N = 2.4 + 1 = 9$, so nine cards suffice.

Topics covered

- The Principle of Inclusion-Exclusion.
- The addition and multiplication rules.
- The Pigeonhole Principle.

Reference

- “Discrete Mathematics with Graph Theory”, Third Edition, E. Goodaire and Michael Parmenter, Pearson Prentice Hall, 2006. pp 184-204.