COT 2104
Homework Assignment 1 (Answers)

1) Classify true or false

a) "$4 \neq 2 + 2$ and $7 < \sqrt{50}$." **False** because one of the two statements is false.
b) "$4 = 2 + 2 \rightarrow 7 < \sqrt{50}$." **True** because both statements are true.
c) "$4 = 2 + 2 \leftrightarrow 7 < \sqrt{50}$." **True** because both statements are true
d) "$4 \neq 2 + 2 \leftrightarrow 7 < \sqrt{50}$." **False** because one of the statements is false while the other is true.
e) "The area of a circle of radius r is $2\pi r$ or its circumference is πr^2." **False** the area of the circle with radius r is not $2\pi r$ and its circumference is not πr^2
f) "$2 + 3 = 5 \rightarrow 5 + 6 = 10$." **False** the hypothesis is true, but the conclusion is false.

2) Write down the negation of each of the following statements in clear and concise English. Do not use the expression It is not the case that" in your answers.

a) x is a real number and $x^2 + 1 = 0$.
x is not a real number or $x^2 + 1 \neq 0$

b) Every integer is divisible by a prime.
There exists an integer which is not divisible by a prime.

c) There exist a, b, and c such that $(ab)c \neq a(bc)$.
$(ab)c = a(bc)$ for a, b, c

d) For every $x > 0$, $x^2 + y^2 > 0$ for all y.
There exists $x > 0$ and some y such that $x^2 + y^2 \leq 0$

e) There exists an infinite set whose proper subsets are all finite.
For any infinite set, some proper subset is not finite.

3) Write the converse and the contrapositive

a) $x^2 = 1 \rightarrow x = \pm 1$.
Converse: $x = \pm 1 \rightarrow x^2 = 1$
Contrapositive: $x \neq 1$ and $x \neq -1 \rightarrow x^2 \neq 1$

b) $ab = 0 \rightarrow a = 0$ or $b = 0$.
Converse: $a = 0$ or $b = 0 \rightarrow ab = 0$
Contrapositive: $a \neq 0$ and $b \neq 0 \rightarrow ab \neq 0$

c) If ΔBAC is a right triangle, then $a^2 = b^2 + c^2$.
Converse: if $a^2 = b^2 + c^2$ then ΔBAC is a right triangle
Contrapositive: if $a^2 \neq b^2 + c^2$ then ΔBAC is not a right triangle,

4) Using "for all" and "there exists" as appropriate.

a) For real x, 2^x is never negative.

$2^x \geq 0$ for all real member x

b) There are infinitely many primes.

For every set of primes $p_1, p_2, \ldots, p_n$, there exist a prime not in this set

c) All positive real numbers have real square roots.

For every real number x > 0, there exists a real number a such that $a^2 = x$

5) What is the hypothesis and what is the conclusion?

a) The square of the length of the hypotenuse of a right-angled triangle is the sum of the squares of the lengths of the other two sides.

Hypothesis: T is a right angled triangle with hypotenuse of length c and the other sides of lengths a and b.
Conclusion: $a^2 + b^2 = c^2$

b) All primes are even.
Hypothesis: p is a prime.
Conclusion: p is even.

6) Determine whether or not the following implication is true.

"x is an even integer $\leftrightarrow$ x + 2 is an even integer."

Let A be the statement "x is an even integer"
Let B be the statement "x + 2 is an even integer"

x = 2k for some other integer k, then x + 2 = 2k + 2 = 2(k + 1) is also twice an integer, so x + 2 is even.

A is true B is true

$A \rightarrow B$ is true $B \rightarrow A$ is true

7) Let n be integer greater than 1 and consider the statement "A: 2^n -1 prime is necessary for n to be prime."

In this case you have two prepositions:
p: n is prime
q: $2^n - 1$ prime

a) Write A as an implication.

N prime $\rightarrow 2^n - 1$ prime

b) Write $\mathcal{A}$ in the form "p is sufficient for q."

n prime is sufficient for $2^n - 1$ to be prime.

c) Write the converse of $\mathcal{A}$ as an implication.

$2^n - 1$ prime $\rightarrow$ n prime.

d) Determine whether the converse of $\mathcal{A}$ is true or false.
Hint: $(2^a)^{b-1} = (2^a - 1)[(2^a)^{b-1} + (2^a)^{b-2} + 2^a + 1]$

The converse of A is true. To show this, we establish the contrapositive. Thus, we assume n is not prime. Then there exists a pair of integers a and b such that a > 1, b > 1, and n = ab. Using the hint, we can factor $2^n - 1$ as

$$\mathbf{2^n - 1 = (2^a)^{b-1} = (2^a - 1)[(2a)^{b-1} + (2^a)^{b-2} + 2^a + 1]}$$

Since a > 1 and b > 1, we have $2^a - 1 > 1$ and $(2^a)^{b-1} + (2^a)^{b-2} + \ldots + 2^a + 1 > 1$, so $2^n - 1$ is the product of two integers of which exceed one. Hence, $2^n - 1$ is not prime.

8) For $a^2 - b^2$ to be odd, it is necessary and sufficient for one of a or b to be even while the other is odd.

I) a, b even.
In this case, a = 2n and b = 2m for some integers m and n, so $a^2 - b^2 = 4n^2 - 4m^2 = 4(n^2 - m^2)$ is even.

II) a, b odd
In this case, a = 2n + 1 and b = 2m + 1 for some integers m and n, so $a^2 - b^2 = (4n^2 + 4n +1) - (4m^2 + 4m + 1) = 4(n^2 + n - m^2 - m)$ is even.

III) a even, b odd
In this case, a = 2n and b = 2m + 1 for some integers m and n, so $a^2 - b^2 = 4n^2 - (4m^2 + 4m + 1) = 4(n^2 + n - m^2 - m) - 1$ is odd.

IV) a odd, b even
In this case, a = 2n + 1 and b = 2mfor some integers m and n, so $a^2 - b^2 = (4n^2 + 4n + 1) - 4m^2 = 4(n^2 + n - m^2) - 1$ is odd.

9) We assert that $x + 1/x \geq 2$ if and only if $x > 0$

($\rightarrow$) We offer a proof by contradiction.
Suppose $x + 1/x \geq 2$ but $x > 0$ is not true; thus $x \leq 0$.
If x = 0, 1 /x is not defined, so x < 0. In this case, however, x + 1/x < 0, a contradiction.

(←) Conversely, assume that $x > 0$. Note that $(x - 1)^2 \geq 0$ implies $x^2 - 2x + 1 \geq 0$
Which in turn implies $x^2 + 1 \geq 2x$.
Division by the positive number x gives $x + 1/x \geq 2$ as required.

10) There exists an integer k such that $n = 4k + 1$ or $n = 4k + 3$

Case 1: $n = 4k + 1$
If k is even, there exits an integer m such that $k = 2m$, so $n = 4(2m) + 1 = 8m + 1$, and the desired conclusion is true.
If k is odd, there exists an integer m such that $k = 2m + 1$, so $n = 4(2m + 1) + 1= 8m + 5$, and the desired conclusion is true.

Case 2: $n = 4k + 3$

If k is even, there exits an integer m such that $k = 2m$, so $n = 4(2m) + 3 = 8m + 3$, and the desired conclusion is true.
If k is odd, there exists an integer m such that $k = 2m + 1$, so $n = 4(2m + 1) + 3= 8m + 7$, and the desired conclusion is true. In all cases, the desired conclusion is true.

11)
a) $(p \wedge q) \vee ((\neg p) \rightarrow q)$

p	**q**	**¬p**	**p ∧ q**	**(¬p) → q)**	**(p ∧ q) ∨ ((¬p) → q)**
T	**T**	**F**	**T**	**T**	**T**
T	**F**	**F**	**F**	**T**	**T**
F	**T**	**T**	**F**	**T**	**T**
F	**F**	**T**	**F**	**F**	**F**

b) $\neg (p \wedge (q \vee p)) \leftrightarrow p$

p	**q**	**p ∨ q**	**p ∧(q ∨ p)**	**¬(p ∧(q ∨ p))**	**¬(p ∧(q ∨ p)) ↔ p**
T	**T**	**T**	**T**	**F**	**F**
T	**F**	**T**	**T**	**F**	**F**
F	**T**	**T**	**F**	**T**	**F**
F	F	F	F	T	F

12) $p \rightarrow q$ is false, determine the truth value of $(p \wedge (\neg q)) \vee ((\neg p) \rightarrow q)$

If $p \rightarrow q$ is false, then p is true and q is false. We work with one row

p	**q**	**¬q**	**(p ∧ (¬q)**	**¬p**	**(¬p) → q**	**(p ∧ (¬q)) ∨ ((¬p) → q)**
T	**F**	**T**	**T**	**F**	**T**	**T**

13) Determine the truth value for $[p \rightarrow (q \wedge (\neg r))] \vee [r \leftrightarrow ((\neg s) \vee q)]$ where p, q, r, and s are all false.

p	q	r	s	$\neg r$	$q \wedge (\neg r)$	$p \rightarrow (q \wedge (\neg r))$	$\neg r$	$(\neg s) \vee q$
F	F	F	F	T	F	T	T	T

$r \leftrightarrow ((\neg s) \vee q)$	$p \rightarrow (q \wedge (\neg r))] \vee [r \leftrightarrow ((\neg s) \vee q)$
F	T

14) Show that $q \rightarrow (p \rightarrow q)$ is a tautology

p	q	$p \rightarrow q$	$q \rightarrow (p \rightarrow q)$
T	T	T	T
T	F	F	T
F	T	T	T
F	F	T	T

∴ This implication is a tautology (the result is true)

15) Determine whether or not the following argument is valid

a. $p \rightarrow q$
$r \rightarrow q$

$r \rightarrow p$

p	q	r	$p \rightarrow q$	$r \rightarrow q$	$r \rightarrow p$
T	T	T	T	T	T
T	F	T	F	F	T
F	T	T	T	T	F
F	F	T	T	F	F
T	T	F	T	T	T
T	F	F	F	T	T
F	T	F	T	T	T
F	F	F	T	T	T

We analyze with a truth table. In row three, the premises are true the conclusion is not. The argument is not valid.

b. $p \rightarrow q$
$(q \vee (\neg r)) \rightarrow (p \wedge s)$

$s \rightarrow (r \vee q)$

This can be solved using a truth table with 16 rows. Alternatively, we can proceed as follow. Assume that the argument is not valid. This means that we can find truth values for p, q, r, and s such that the premises are true but the conclusion is false. Since $s \rightarrow (r \vee q)$ is false, we must have s true and $r \vee q$ false. But this means both r and q are false. Since $p \rightarrow q$ is true and q is false, p must be false. But then $q \vee (\neg r)$ is true and $p \wedge s$ is false, contradicting the truth of $(q \vee (\neg r)) \rightarrow (p \wedge s)$. Hence we have a contradiction, so the argument is valid.

NOTE: Some exercises could have different variants of correct answers