MAC2312 Test 3 Name Key

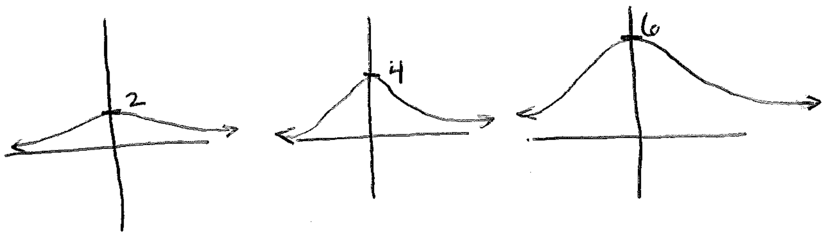
D. Howard 3-16

Reach

1. Sketch a graph of the family of curves called the witch of Maria Agnesi.
$x = 2a\cot\theta \quad y = 2a\sin^2\theta$. Show how the graphs change when the value of "a" increases?

parametric mode
radians mode
window
$t \in [0, 2\pi] \times 0.1$
$x \in [-10, 10] \times 1$
$y \in [-10, 10] \times 1$

$a = 1$ $a = 2$ $a = 3$

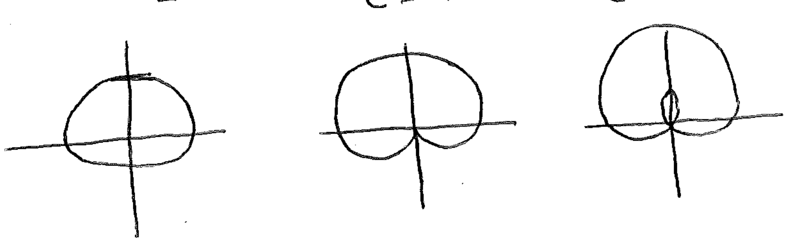


overall shape doesn't change, but the maximum increases.

2. Sketch the family of curves of $r = 1 + c\sin\theta$ for $c = \frac{1}{2}$, $c = 1$, and $c = 2$.

polar mode
radians mode
window
$\theta \in [0, 2\pi] \times 0.1$
$x \in [-3, 3] \times 1$
$y \in [-3, 3] \times 1$

$c = \frac{1}{2}$ $c = 1$ $c = 2$



shape changes.

$y = \ln(\sec x)$

3. Find the exact length of the curve $= \ln(secx), 0 \le x \le \frac{\pi}{4}$.

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx \qquad y' = \frac{1}{\cancel{\sec x}} \cdot \cancel{\sec x} \tan x$$

$$y' = \tan x$$

$$L = \int_0^{\pi/4} \sqrt{1 + (\tan x)^2}\, dx$$

$$L = \int_0^{\pi/4} \sqrt{\sec^2 x}\, dx$$

$$L = \int_0^{\pi/4} \sec x\, dx$$

$$L = \ln|\sec x + \tan x| \Big/_0^{\pi/4}$$

$$L = \ln\left|\sec\left(\tfrac{\pi}{4}\right) + \tan\left(\tfrac{\pi}{4}\right)\right| - \ln|\sec(0) + \tan(0)|$$

$$L = \ln\left|\tfrac{2}{\sqrt{2}} + 1\right| - \ln|1 + 0|$$

$$\boxed{L = \ln\left|\tfrac{2}{\sqrt{2}} + 1\right|}$$

4. Find the exact surface area for the given curve rotated about the y-axis.

$x = \sqrt{a^2 - y^2}, 0 \le y \le \frac{a}{2}$

$$SA = \int_a^b 2\pi f(y) \sqrt{1 + [f'(y)]^2}\, dy \qquad \text{OR} \qquad \int_a^b 2\pi x \sqrt{1 + [f'(x)]^2}\, dx$$

But everything is given with respect to y. So Use other form.

$$SA = \int_0^{a/2} 2\pi \sqrt{a^2 - y^2} \sqrt{1 + \left[\frac{1}{\cancel{2}\sqrt{a^2 - y^2}} \cdot \cancel{2}y\right]^2}\, dy$$

$$SA = \int_0^{a/2} 2\pi \sqrt{a^2 - y^2} \sqrt{1 + \frac{y^2}{a^2 - y^2}}\, dy$$

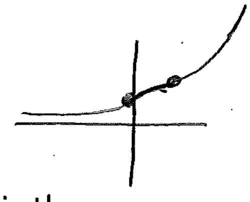
$$SA = \int_0^{a/2} 2\pi \sqrt{a^2 - y^2} \sqrt{\frac{a^2 - \cancel{y^2} + \cancel{y^2}}{a^2 - y^2}}\, dy$$

$$SA = \int_0^{a/2} 2\pi \sqrt{a^2 - y^2} \sqrt{\frac{a^2}{a^2 - y^2}}\, dy$$

$$SA = \int_0^{a/2} 2\pi \cancel{\sqrt{a^2 - y^2}} \frac{a}{\cancel{\sqrt{a^2 - y^2}}}\, dy$$

$$SA = \int_0^{a/2} 2\pi a\, dy = 2\pi a y \Big/_0^{a/2} = 2\pi a\left(\frac{a}{2} - 0\right) = \frac{\cancel{2}\pi a^2}{\cancel{2}}$$

$$\boxed{SA = \pi a^2}$$



5. Calculate the first and second derivative. For which values of t is the curve concave up?

$x = t^2 + 1, y = e^t - 1$

$x' = 2t$

$y' = e^t$

$$\frac{dy}{dx} = \frac{y'}{x'} = \frac{e^t}{2t}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{y'}{x'}\right)}{x'} = \frac{\frac{2te^t - e^t \cdot 2}{(2t)^2}}{2t} = \frac{2e^t(t-1)}{4t^2 \cdot 2t}$$

$$\frac{d^2y}{dx^2} = \frac{\cancel{2}e^t(t-1)}{4\cancel{8}t^3} = \frac{e^t(t-1)}{4t^3}$$

$$\boxed{\begin{array}{l} y'(x) = \dfrac{e^t}{2t} \\ y''(x) = \dfrac{e^t(t-1)}{4t^3} \\ \text{Concave up } (-\infty, 0) \\ \qquad \cup (1, \infty) \end{array}}$$

$\frac{e^t(t-1)}{4t^3} = 0$

$e^t(t-1) = 0$

$e^t = 0 \quad t-1=0$

$t = \emptyset \quad t = 1$

$\frac{e^t(t-1)}{4t^3} = \emptyset$

$4t^3 = 0$

$t = 0$

$++++ \quad | \quad --- \quad | \quad ++++$

$t=0 \quad t=1$

$y''(x) = \frac{e^t(t-1)}{4t^3}$

6. Calculate the area of the region that lies inside the first curve and outside the second curve.

$r^2 = 8\cos(2\theta), r = 2$

$r = \sqrt{8\cos(2\theta)} \qquad r = 2$

Polar mode

radians mode

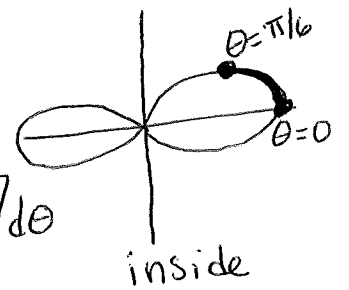
window

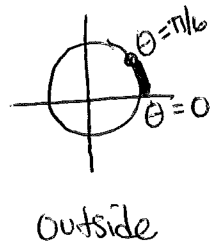
$\theta \in [0, 10\pi] \times 0.1$

$x \in [-3, 3] \times 1$

$y \in [-3, 3] \times 1$

$$A = \int_a^b \frac{1}{2} r^2 \, d\theta$$





$$A = 4\int_0^{\pi/6} \frac{1}{2}\left[\left(\sqrt{8\cos 2\theta}\right)^2 - 2^2\right] d\theta$$

$$A = 2\int_0^{\pi/6} (8\cos 2\theta - 4)\, d\theta$$

$$A = 2\left[\frac{8\sin 2\theta}{2} - 4\theta\right]\Big|_0^{\pi/6}$$

$$A = 2\left[4\sin 2\theta - 4\theta\right]\Big|_0^{\pi/6}$$

$$A = 8(\sin 2\theta - \theta)\Big|_0^{\pi/6}$$

$$A = 8\left(\frac{\sqrt{3}}{2} - \frac{\pi}{6} - (0 - 0)\right)$$

$$\boxed{A = 4\left(\sqrt{3} - \frac{\pi}{3}\right)}$$

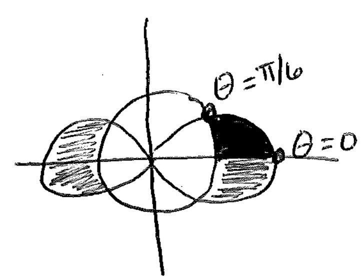
$2 = \sqrt{8\cos(2\theta)}$

$4 = 8\cos(2\theta)$

$\frac{1}{2} = \cos(2\theta)$

$\frac{\pi}{3} = 2\theta$

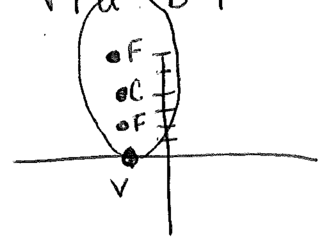
$\frac{\pi}{6} = \theta$



$h \quad k$

7. Find the equation of an ellipse with center (-1, 4), vertex (-1, 0), and focus (-1,6),

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \qquad c = \sqrt{|a^2-b^2|}$$



$$\frac{(x+1)^2}{a^2} + \frac{(y-4)^2}{b^2} = 1$$

$$\boxed{\frac{(x+1)^2}{12} + \frac{(y-4)^2}{16} = 1}$$

$b = 4$ = distance from center to vertex

$c = 2$ = distance from center to focus

$2 = \sqrt{|a^2 - 4^2|}$ $\qquad b > a$

$2 = \sqrt{|a^2 - 16|}$

$2 = \sqrt{16 - a^2}$

$4 = 16 - a^2$

$12 = a^2$

8. Find the eccentricity and identify the conic.

a. $r = \frac{12}{3-10cos\theta} = \frac{12}{3(1-\frac{10}{3}\cos\theta)}$ $\qquad e = \frac{10}{3} > 1$ hyperbola

b. $r = \frac{11}{3+3sin\theta} = \frac{11}{3(1+1\sin\theta)}$ $\qquad e = 1$ parabola

c. $r = \frac{2}{3+2cos\theta} = \frac{2}{3(1+\frac{2}{3}\cos\theta)}$ $\qquad e = \frac{2}{3}$ ellipse