4.3 Quadratic Functions and Their Properties

Learning Objectives

1. Identify the Vertex and Axis of Symmetry of a Quadratic Function
2. Graph a Quadratic Function Using Its Vertex, Axis, and Intercepts
3. Find a Quadratic Function Given Its Vertex and One Other Point
4. Find the Maximum or Minimum Value of a Quadratic Function

Definition 1

A **quadratic function** is a function of the form

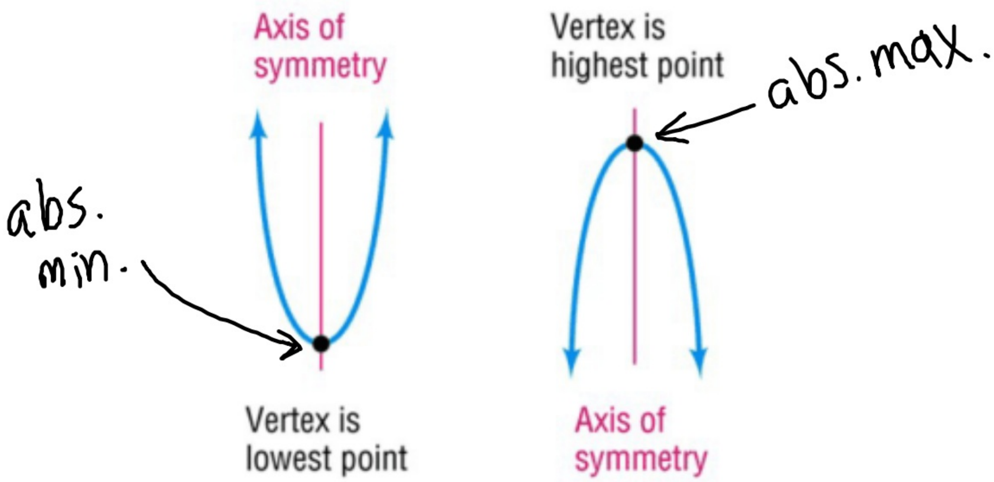
$$f(x) = ax^2 + bx + c$$

where *a*, *b*, and *c* are real numbers and $a \neq 0$.

The domain of a quadratic function is the set of all real numbers.

Graphs of a Quadratic Function

Figure 16



(a) Opens up

$a > 0$

(b) Opens down

$a < 0$

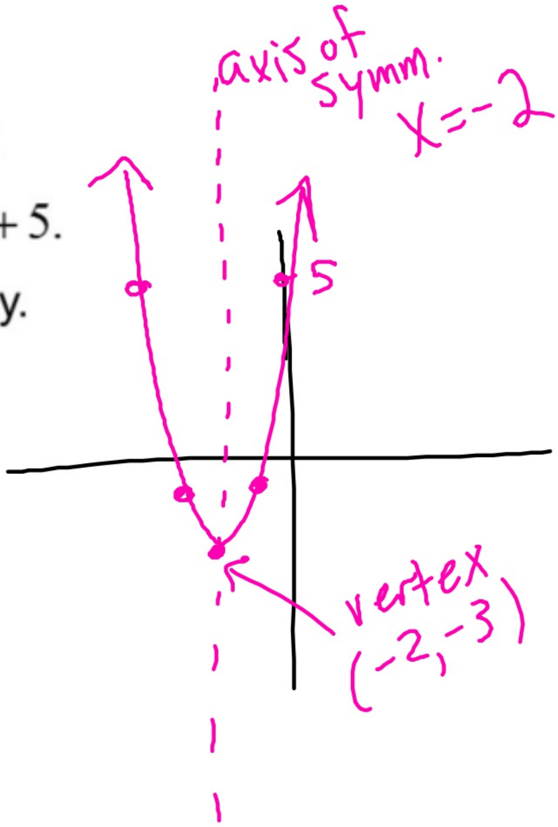
Example 1

Graphing a Quadratic Function

Graph the function $f(x) = 2x^2 + 8x + 5$.

Find the vertex and axis of symmetry.

x	$f(x)$
-4	5
-3	-1
-2	-3
-1	-1
0	5



Properties of the Graph of a Quadratic Function

$f(x) = ax^2 + bx + c \quad a \neq 0$

x-coor. of vertex $= -\dfrac{b}{2a}$

y-coor. plug in x

* Vertex $= \left(-\dfrac{b}{2a}, f\left(-\dfrac{b}{2a} \right) \right)$

* Axis of symmetry : the vertical line $x = -\dfrac{b}{2a}$

Parabola opens up if $a > 0$; the vertex is a minimum point.

Parabola opens down if $a < 0$; the vertex is a maximum point

Example 2

Locating the Vertex without Graphing

Without graphing, locate the vertex and axis of symmetry of the parabola defined by $f(x) = -3x^2 + 6x + 1.$

Does it open up or down?

axis of sym
$x = 1$

vertex
$(1, 4)$

vertex: $x\text{-coor.} = \dfrac{-b}{2a} = \dfrac{-6}{2(-3)} = 1$

$y\text{-coor} = f(1) = -3(1)^2 + 6(1) + 1 = 4$

Quadratic Formula

If $ax^2+bx+c=0$

then $x=\dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

Solving $ax^2+bx+c=0$ gives you the <u>x-intercepts</u> of the parabola

Example 3

Graphing a Quadratic Function Using Its Vertex, Axis, and Intercepts

(a) Use the information from Example 2 and the locations of the intercepts to graph $f(x) = -3x^2 + 6x + 1$. vertex: $(1,4)$

(b) Determine the domain and the range of *f*.

(c) Determine where *f* is increasing and where it is decreasing.

(b) D: $(-\infty, \infty)$
R: $(-\infty, 4]$

(c) increasing on $(-\infty, 1)$
decreasing on $(1, \infty)$

$f(x) = -3x^2 + 6x + 1$

x-ints: Solve $-3x^2 + 6x + 1 = 0$

$$x = \frac{-6 \pm \sqrt{36 - 4(-3)(1)}}{2(-3)} = \frac{-6 \pm \sqrt{48}}{-6}$$

$$\approx \frac{-6 \pm 6.9}{-6} \rightarrow \frac{-6 + 6.9}{-6} \rightarrow \frac{0.9}{-6} \rightarrow -0.15$$

$$\frac{-6 - 6.9}{6} \rightarrow \frac{-12.9}{-6} \rightarrow 2.15$$

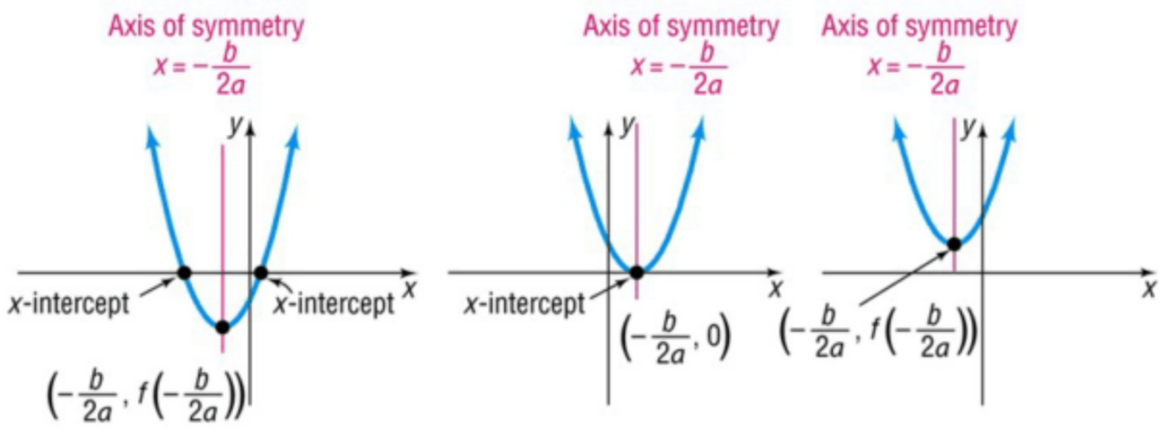
$\boxed{\text{x-ints: } (-0.15, 0) \text{ and } (2.15, 0)}$

(These are approximates of the x-int.'s to two decimal place. These are NOT exact!)

The *x*-Intercepts of a Quadratic Function

1. If the discriminant $b^2 - 4ac > 0$, the graph of $f(x) = ax^2 + bx + c$ has two distinct *x*-intercepts so it crosses the *x*-axis in two places.
2. If the discriminant $b^2 - 4ac = 0$, the graph of $f(x) = ax^2 + bx + c$ has one *x*-intercept so it touches the *x*-axis at its vertex.
3. If the discriminant $b^2 - 4ac < 0$, the graph of $f(x) = ax^2 + bx + c$ has no *x*-intercepts so it does not cross or touch the *x*-axis.

Figure: f(x) = ax^2 + b x + c, a > 0



(a) $b^2 - 4ac > 0$
Two x-intercepts

(b) $b^2 - 4ac = 0$
One x-intercept
(vertex)

(c) $b^2 - 4ac < 0$
No x-intercepts

Example: $f(x) = x^2 - 6x + 9$

Use the discriminant to determine the number ~~and type~~ of solutions for the quadratic function given.

$b^2 - 4ac$

$(-6)^2 - 4(1)(9)$

$36 - 36$

0

1 x-int.

Example: $f(x) = 2x^2 + x + 1$

Use the discriminant to determine the number ~~and type~~ of solutions for the quadratic function given.

$b^2 - 4ac$

$1^2 - 4(2)(1)$

$1 - 8$

-7

No x-int.'s

Definition 3

If the vertex (h, k) and one additional point on the graph of a quadratic function $\boxed{f(x) = ax^2 + bx + c,}$ $a \neq 0$, are known, then

standard form

$$\boxed{f(x) = a(x - \underline{h})^2 + \underline{k}}$$

vertex form

can be used to obtain the quadratic function.

$f(x) = y$

Example 4

Finding the Quadratic Function Given Its Vertex and One Other Point

Determine the quadratic function whose vertex is $(1, -5)$ and whose y-intercept is -3. … $(0, -3)$

h k

x y

$$f(x) = a(x-h)^2 + k$$

First, sub. in h,k,x,y to solve for a.

$$-3 = a(0-1)^2 - 5 \longrightarrow -3 = 1a - 5 \rightarrow a = 2$$

Next, sub. h, k, a into vertex form.

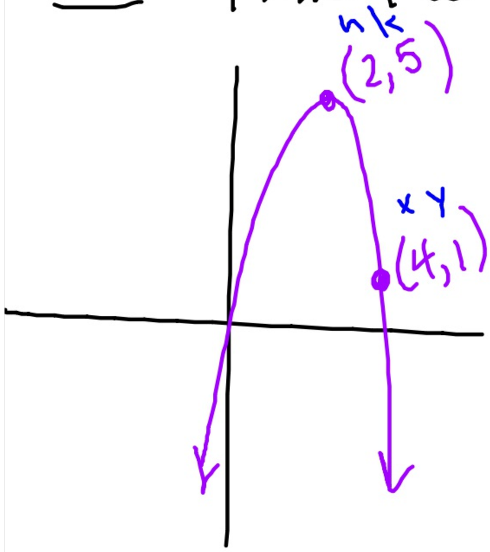
$$f(x) = a(x-h)^2 + k$$

$$\boxed{f(x) = 2(x-1)^2 - 5}$$ vertex form

$$f(x) = 2(x-1)(x-1) - 5$$
$$= 2(x^2 - 2x + 1) - 5$$
$$= 2x^2 - 4x + 2 - 5$$

$$\boxed{f(x) = 2x^2 - 4x - 3}$$ standard form

Ex Find the quadratic function for the parabola



$$y = a(x-h)^2 + k$$

$$1 = a(4-2)^2 + 5$$

$$1 = 4a + 5$$

$$-4 = 4a$$

$$-1 = a$$

$$y = -1(x-2)^2 + 5$$

$$y = -1(x^2 - 4x + 4) + 5$$

$$\boxed{y = -x^2 + 4x + 1}$$

Example 5

Finding the Maximum or Minimum Value of a Quadratic Function

Determine whether the quadratic function

$$f(x) = x^2 - 4x - 5$$

has a maximum or a minimum value. Then find the maximum or minimum value.

$h = \frac{-b}{2a} = \frac{4}{2} = 2$

$k = f(2) = -9$

It has a minimum value. The min. value is $\underline{-9}$.

(located at $x = 2$)