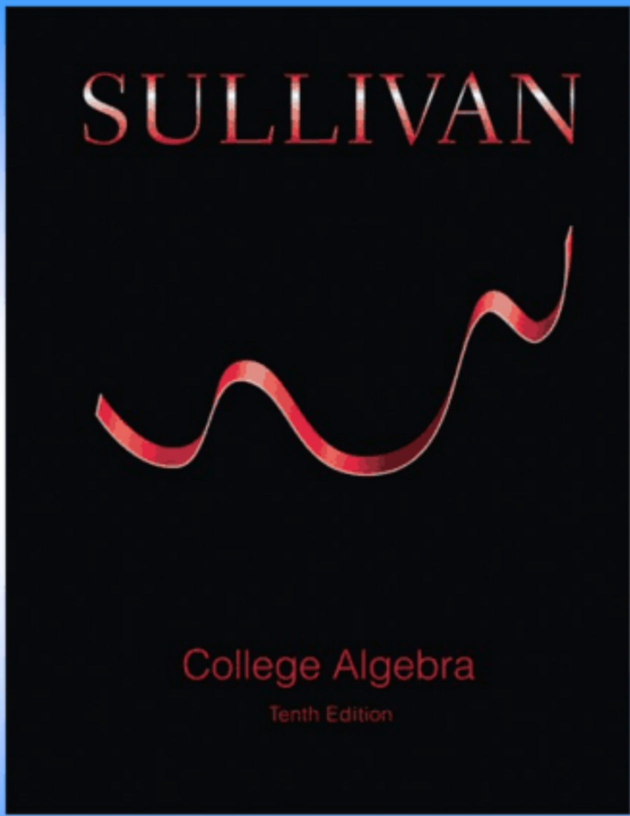
Chapter 5

Section 1

Polynomials



Definition

A **polynomial function** in one variable is a function of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots a_2 x^2 + a_1 x^1 + a_0 x^0 \qquad \textbf{(1)}$$

where $a_n, a_{n-1}, \ldots, a_1, a_0$ are constants, called the **coefficients** of the polynomial, $n \geq 0$ is an integer, and x is the variable. If $a_n \neq 0$, it is called the **leading coefficient**, and n is the **degree** of the polynomial.

The domain of a polynomial function is the set of all real numbers.

→ a's are coefficients ; must be real numbers

→ n is the degree ; must be a whole number : $\{0, 1, 2, 3, \ldots\}$

→ a_n is the leading coeff. ; $a_n x^n$ is leading term → a_0 is the constant (y-int.)

Example 1

Identifying Polynomial Functions

Determine which of the following are polynomial functions. For those that are, state the degree; for those that are not, tell why not. Write each polynomial in standard form, and then identify the leading term and the constant term.

(a) $p(x) = 5x^3 - \frac{1}{4}x^2 - 9$ (b) $f(x) = x + 2 - 3x^4$ (c) $g(x) = \sqrt{x}$ $= x^{\frac{1}{2}}$

(d) $h(x) = \dfrac{x^2 - 2}{x^3 - 1}$ (e) $G(x) = 8$ (f) $H(x) = -2x^3(x - 1)^2$

(a) $p(x) = 5x^3 - \frac{1}{4}x^2 - 9$
yes, 3rd degree polynomial
Leading: $5x^3$
const: -9

(b) $f(x) = -3x^4 + x + 2$
yes, 4th degree
leading: $-3x^4$
Const: 2

(c) Not a polynomial, exponent is not whole

d) $h(x) = \dfrac{x^2 - 2}{x^3 - 1}$

Not a polynomial,
exponent is negative

e) $G(x) = 8x^0$

Yes, zero degree

lead: 8

const: 8

f) $H(x) = -2x^3(x-1)^2$

$H(x) = -2x^3(x-1)(x-1)$

$= -2x^3(x^2 - 2x + 1)$

$= -2x^5 + 4x^4 - 2x^3$

yes, 5th degree

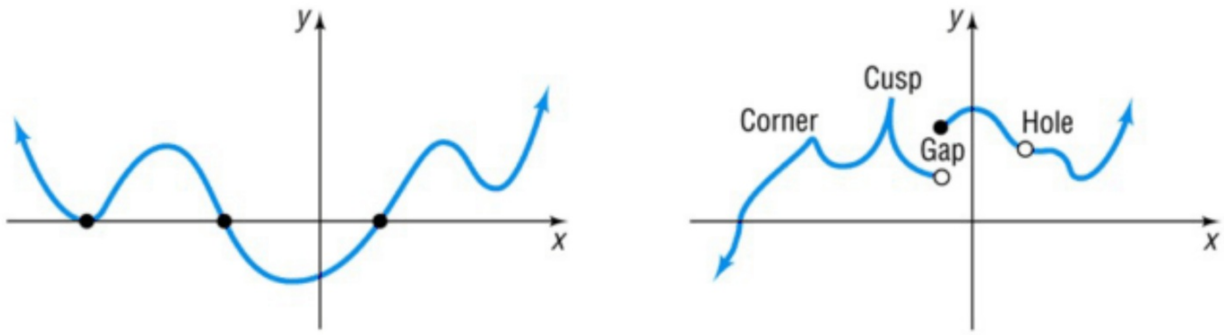
leading term: $-2x^5$

const: 0

Table

Degree	Form	Name	Graph
No degree	$f(x) = 0$	Zero function	The x-axis
0	$f(x) = a_0, \quad a_0 \neq 0$	Constant function	Horizontal line with y-intercept a_0
1	$f(x) = a_1x + a_0, \quad a_1 \neq 0$	Linear function	Nonvertical, nonhorizontal line with slope a_1 and y-intercept a_0
2	$f(x) = a_2x^2 + a_1x + a_0, \quad a_2 \neq 0$	Quadratic function	Parabola: graph opens up if $a_2 > 0$; graph opens down if $a_2 < 0$

Figure



(a) Graph of a polynomial function: smooth, continuous

(b) Cannot be the graph of a polynomial function

Definition

A **power function of degree *n*** is a monomial function of the form

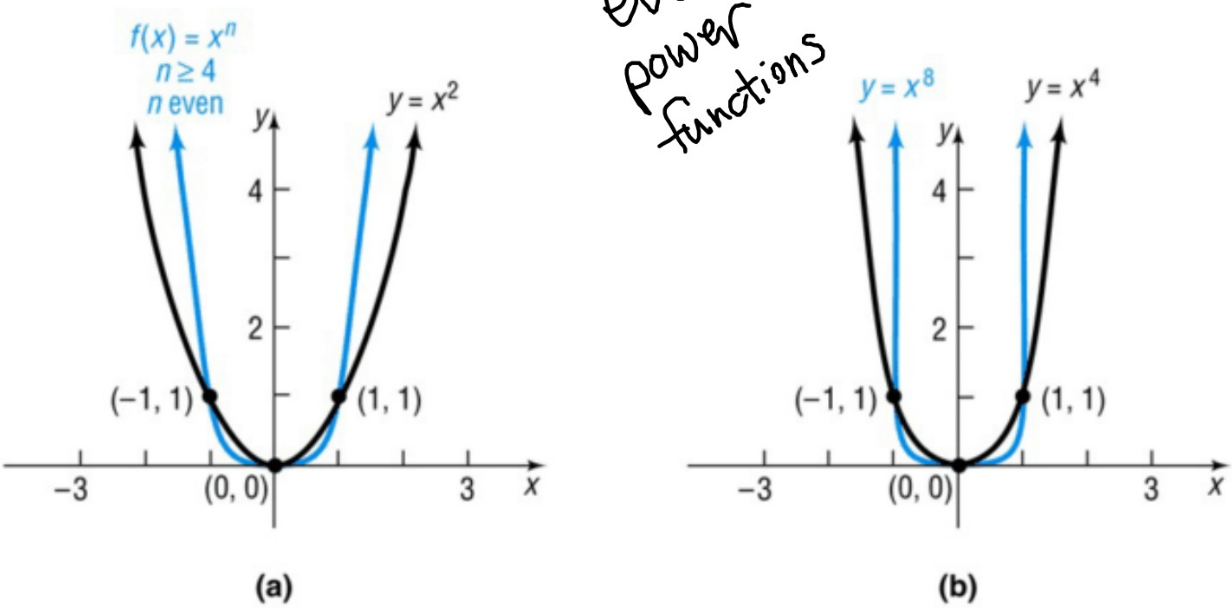
$$f(x) = ax^n \qquad \textbf{(2)}$$

where a is a real number, $a \neq 0$, and $n > 0$ is an integer.

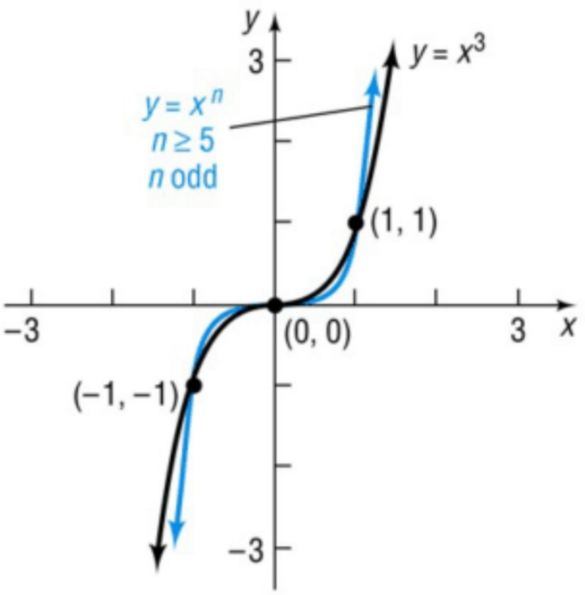
Figure

$y = \underline{\underline{a}}x^n$

even power functions



Figure



odd power functions

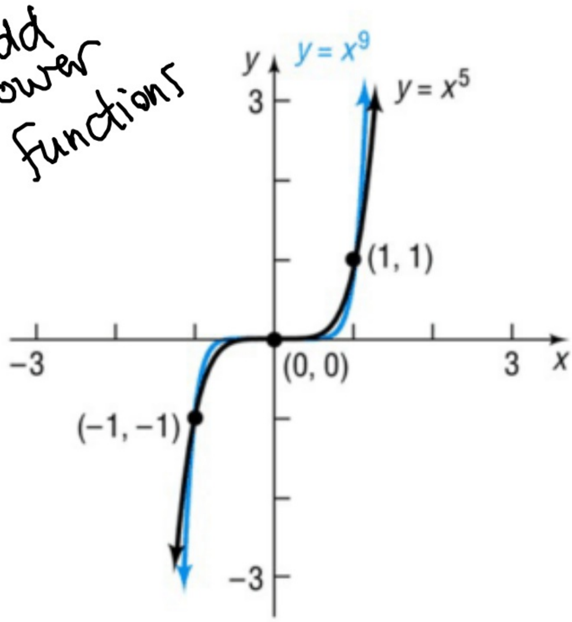
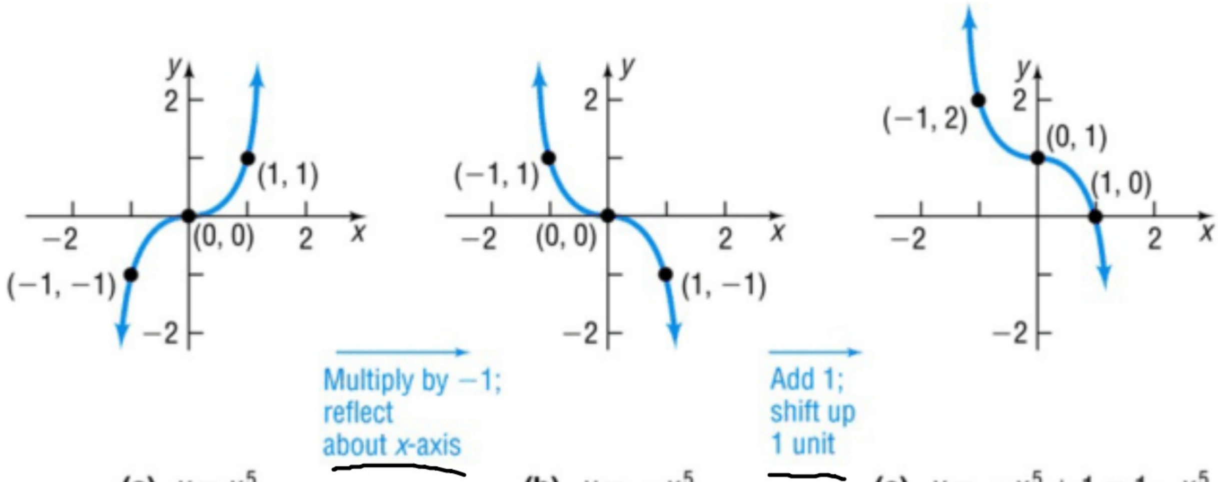


Figure: Graph of $f(x) = 1 - x^5$



(a) $y = x^5$ (b) $y = -x^5$ (c) $y = -x^5 + 1 = 1 - x^5$

Example 2

Graphing a Polynomial Function Using Transformations

Graph: $f(x) = \frac{1}{2}(x - 1)^4$

(solution on next page)

Solution

Figure 8 shows the required stages.

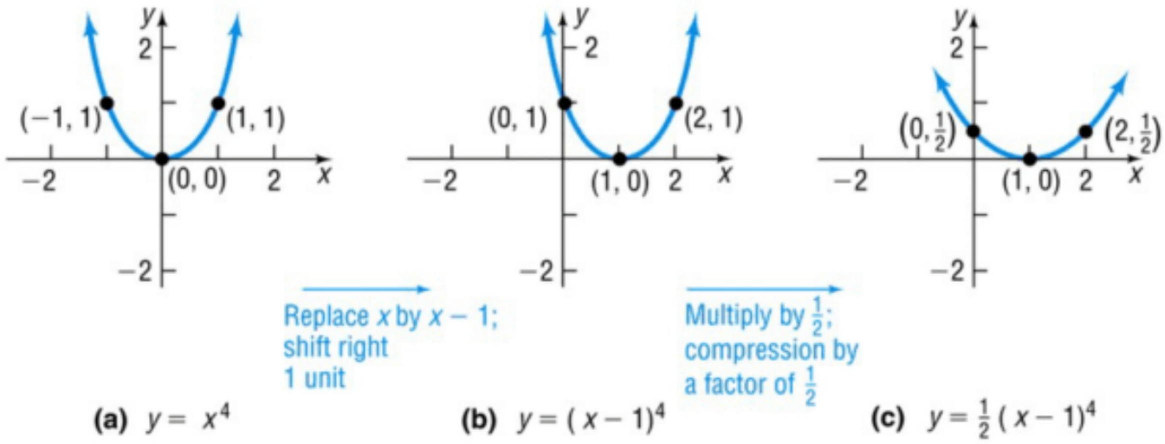
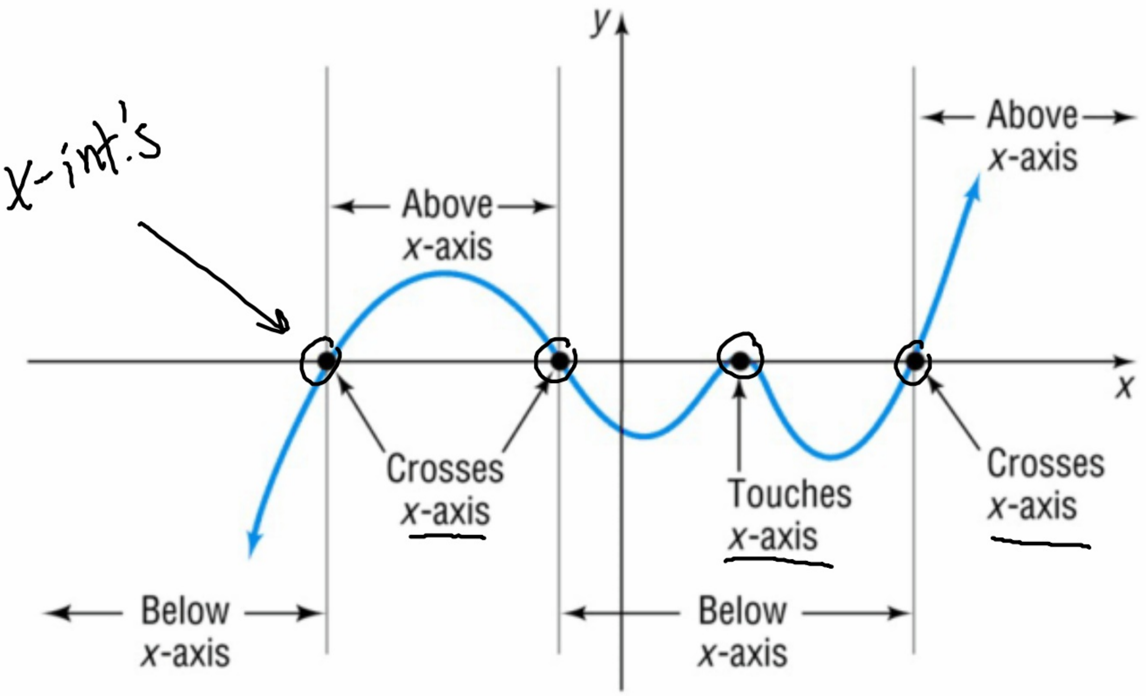


Figure 8

Graph of a Polynomial Function



Definition → x-intercepts / solutions / real zeros

If f is a function and r is a real number for which $f(r) = 0$, then r is called a **real zero** of f.

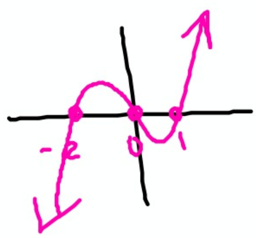
The following statements are equivalent:

1. r is a real zero of a polynomial function f.
2. r is an x-intercept of the graph of f.
3. $x - r$ is a factor of f.
4. r is a solution to the equation $f(x) = 0$.

#3. If r is a real zero of polynomial, then $x-r$ is a factor of the polynomial.

Example 3: Finding Zeros

Find the real zeros of the polynomial function
$f(x) = x^3 + x^2 - 2x$



$x^3 + x^2 - 2x = 0$

$x(x^2 + x - 2) = 0$

$x(x-1)(x+2) = 0$

$x = 0$

$x - 1 = 0$

$x + 2 = 0$

$x = \{0, 1, -2\}$

Definition

Ex: $f(x) = (x-3)^4$

The zero of 3 has a mult. of 4

If $(x - r)^m$ is a factor of a polynomial f and $(x - r)^{m+1}$ is not a factor of f, then r is called a **zero of multiplicity m of f**.* → repeated factors

If r Is a Zero of Even Multiplicity

Numerically: The sign of $f(x)$ does not change from one side to the other side of r.

Graphically: The graph of f **touches** the x-axis at r.

If r Is a Zero of Odd Multiplicity

Numerically: The sign of $f(x)$ changes from one side to the other side of r.

Graphically: The graph of f **crosses** the x-axis at r.

Theorem

Turning Points (max. and min. points)

If f is a polynomial function of degree n, then the graph of f has at most $n - 1$ turning points.

If the graph of a polynomial function f has $n - 1$ turning points, then the degree of f is at least n.

A graph with n-1 turns must be at least an nth degree polynomial.

Example 4

Identifying the Graph of a Polynomial Function

Which of the graphs in Figure 13 could be the graph of a polynomial function? For those that could, list the real zeros and state the least degree the polynomial can have. For those that could not, say why not.

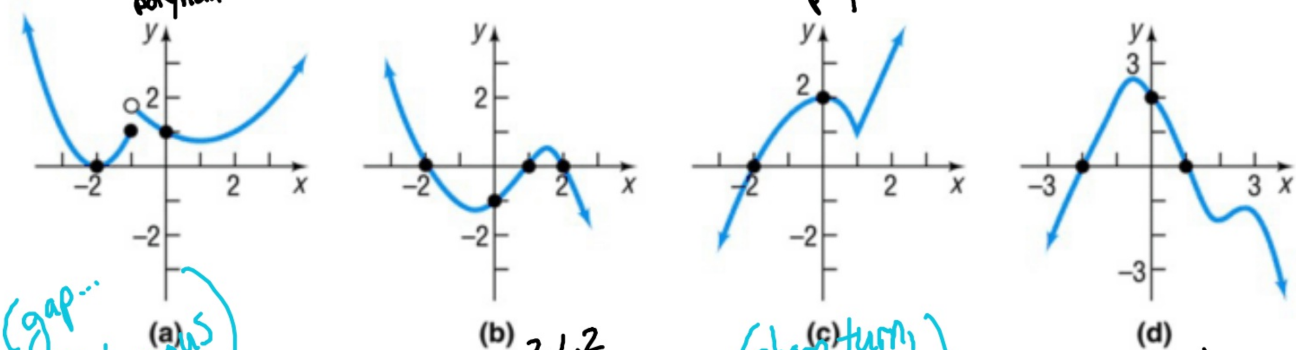


Figure 13

Not a polynomial

(gap... not continuous)

zeros: −2, 1, 2
deg: 3 (could)

Not a polynomial

(sharp turn, not smooth)

zeros: −2, 1
deg: 4 (even)

Theorem

End Behavior

For large values of x, either positive or negative, the graph of the polynomial function

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \quad a_n \neq 0$$

resembles the graph of the power function

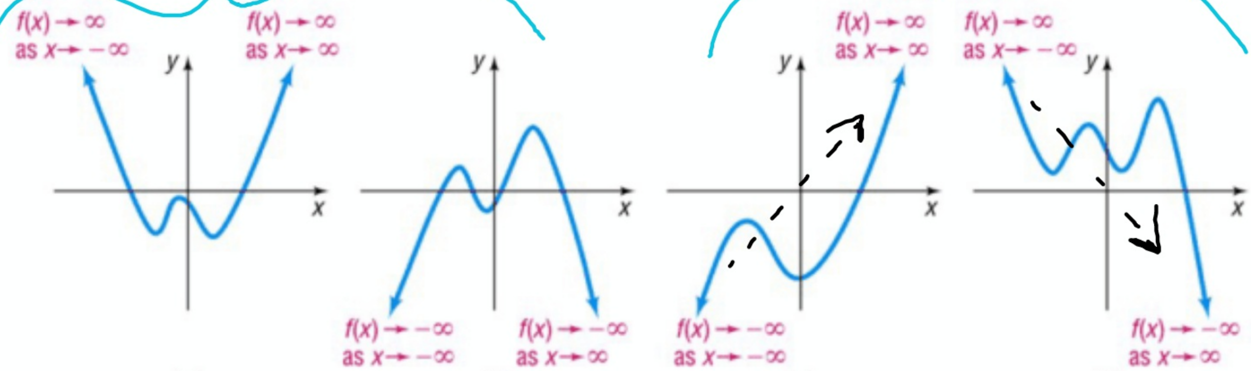
$$y = a_n x^n$$

Figure: End behavior

$f(x) = a_n x^n + \ldots$

both ends go the same direction

ends go opposite directions



(a) $n \geq 2$ even; $a_n > 0$

(b) $n \geq 2$ even; $a_n < 0$

(c) $n \geq 3$ odd; $a_n > 0$

(d) $n \geq 3$ odd; $a_n < 0$

n is even
$a_n > 0$

n is even
$a_n < 0$

n is odd
$a_n > 0$

n is odd
$a_n < 0$

Example 5

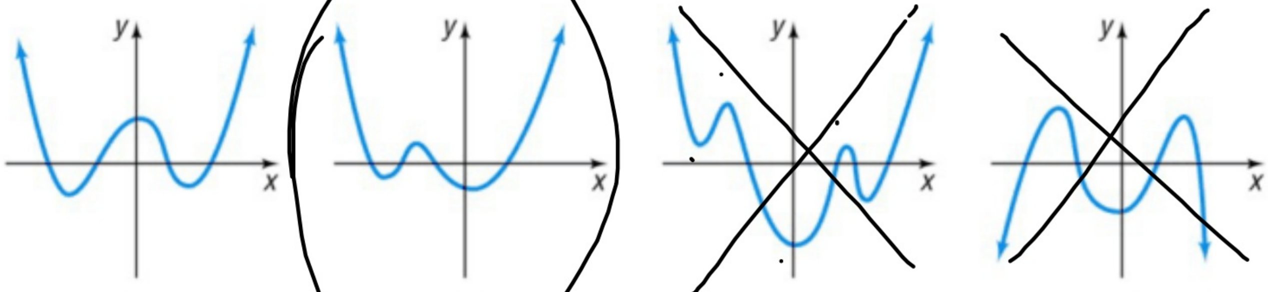
both ends up $\begin{cases} \text{even deg.} \\ a_n > 0 \end{cases}$

Identifying the Graph of a Polynomial Function

Which of the graphs in Figure 16 could be the graph of

$$f(x) = x^4 + ax^3 + bx^2 - 5x - \underline{\underline{6}} \leftarrow \text{y-int. is negative}$$

where $a > 0$, $b > 0$?



(a) (b) (c) (d)

Figure 16

Example 6

Writing a Polynomial Function from Its Graph

Write a polynomial function whose graph is shown in Figure 17 (use the smallest degree possible).

$n = 4$
because it has 3 turns

must have even multiplicity because the graph "touches" at -2

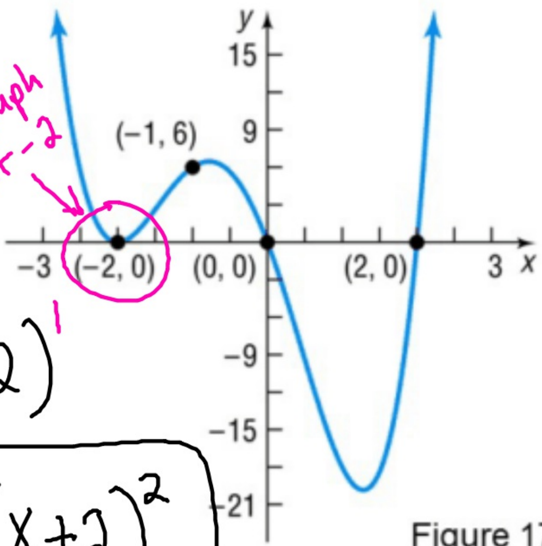


Figure 17

$f(x) =$
$(x+2)^2(x-0)^1(x-2)^1$

$$\boxed{f(x) = x(x-2)(x+2)^2}$$

x-int's:
$-2, 0, 2$
-2: even mult.
$0, 2$: odd mult.

Example 7: Use Given Zeros to Write and Analyze Polynomial Functions

Form a polynomial of degree 3 with zeros $-3, 3,$ and 7. Write your answer in factored form and standard form with a leading coefficient of 1.

$$f(x) = (x+3)(x-3)(x-7)$$

Example 8

How to Analyze the Graph of a Polynomial Function

Analyze the factored form of the polynomial function
$f(x) = (2x-1)(2x+1)(x+3)^2$ $= (2x-1)(2x+1)(x+3)(x+3)$
$= 4x^4 + \ldots - 9$

Step 1: Determine the end behavior.
Step 2: Find the x and y intercepts.
Step 3: Determine the zeros and their multiplicity. Use this to determine if the graph touches the x-axis or if it crosses.
Step 4: Determine the maximum number of turning points.

$$f(x) = (2x - 1)(2x + 1)(x + 3)^2$$

Step 1: Determine the end behavior.

online: "resembles the power function $y = 4x^4$"

which means both ends go up

Step 2: Find the x and y intercepts.

x-ints: $2x - 1 = 0$, $2x + 1 = 0$, $x + 3 = 0$

$x = 1/2, -1/2, -3 \rightarrow (\pm\frac{1}{2}, 0), (-3, 0)$

y-int: $(0, -9)$

$$f(x) = (2x - 1)(2x + 1)(x + 3)^2$$

x-ints!

Step 3: Determine the zeros and their multiplicity. Use this to determine if the graph touches the x-axis or if it crosses.

$x = -\frac{1}{2}$ mult. of 1

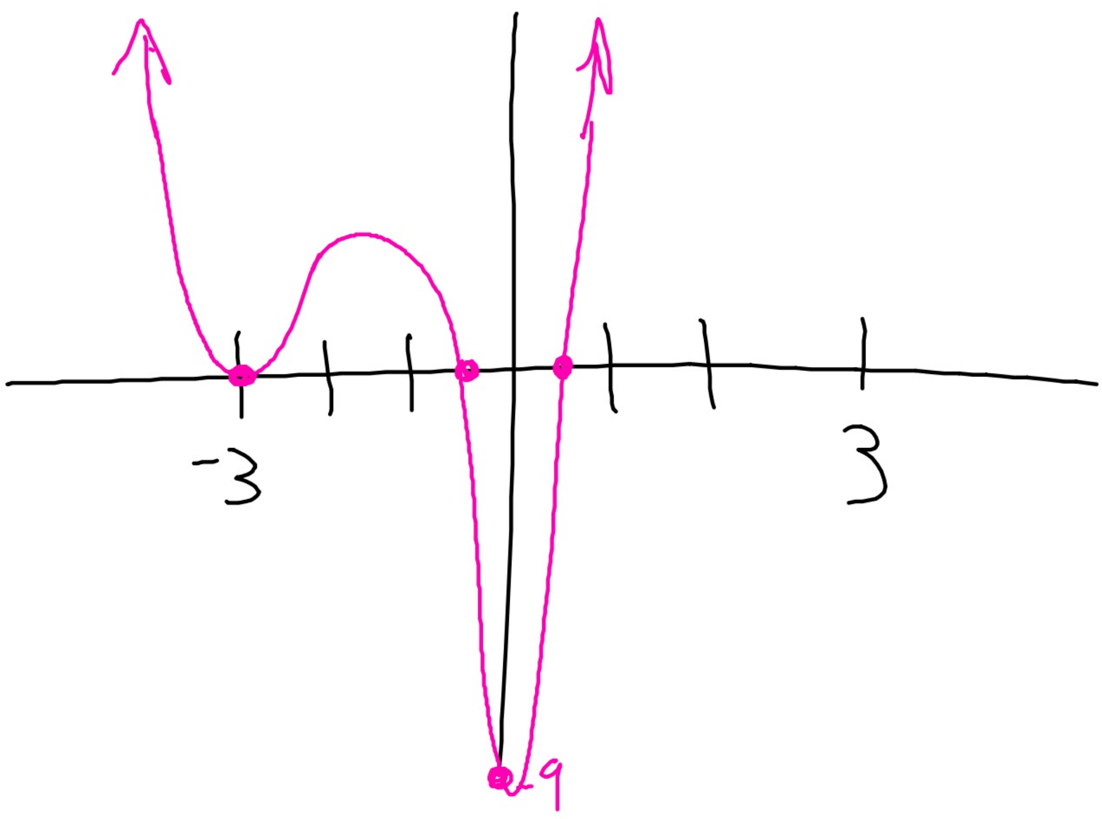
$x = \frac{1}{2}$ mult. of 1

cross the x-axis at $\pm \frac{1}{2}$

$x = -3$ mult. of 2 → touch the x-axis at -3

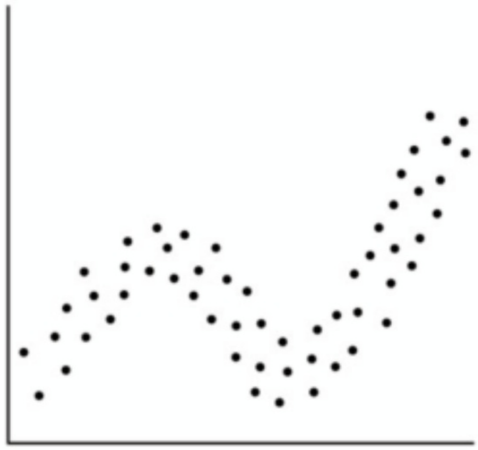
Step 4: Determine the maximum number of turning points.

$(n-1)$ max. of 3 turns



-3
3
-9

Figure: Cubic relation



$y = ax^3 + bx^2 + cx + d, a > 0$

(a)

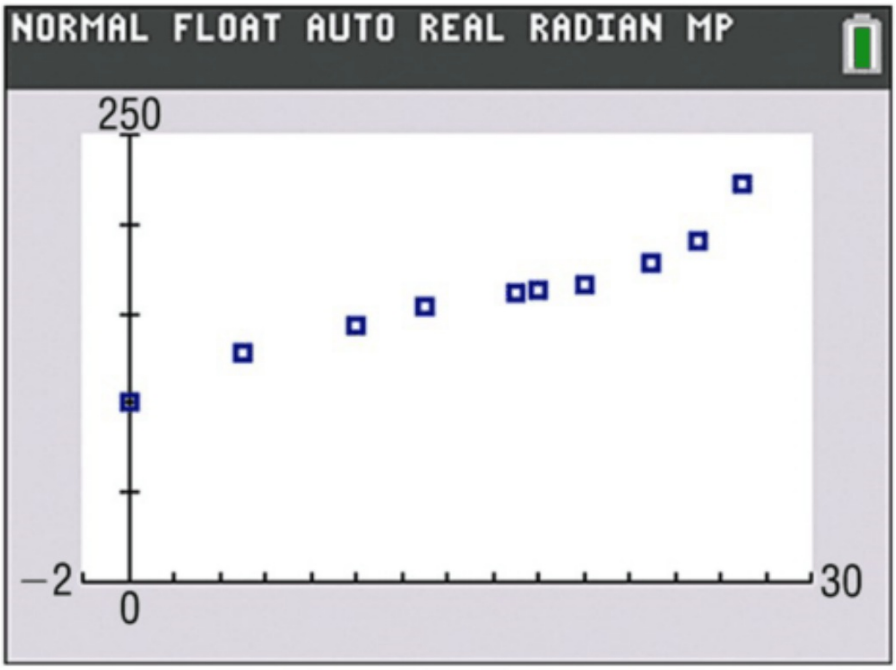
$y = ax^3 + bx^2 + cx + d, a < 0$

(b)

Example – Find the Model

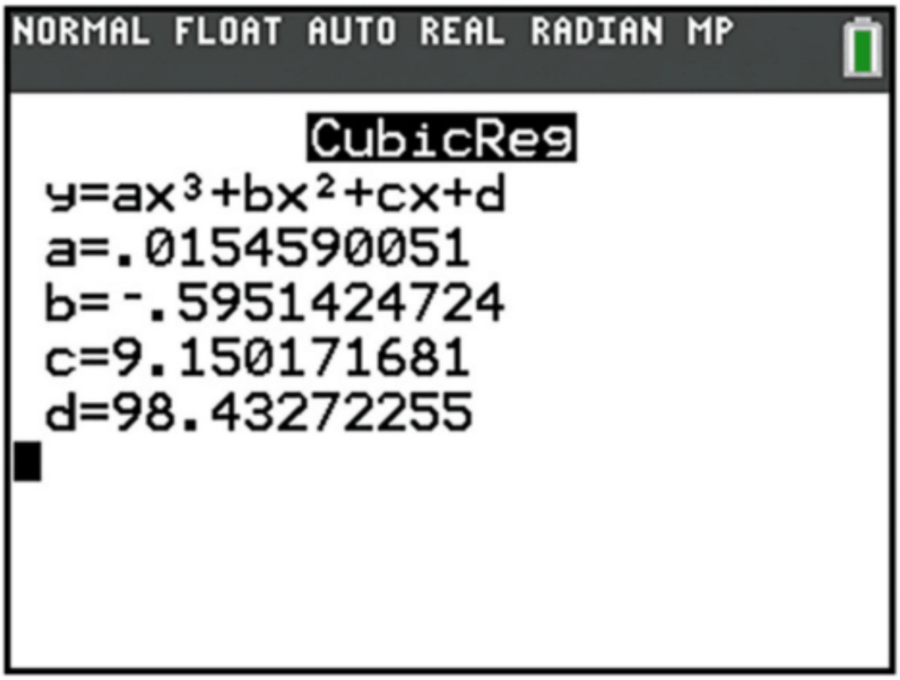
Number *x* of Textbooks, (thousands)	Cost, *C* ($1000s)
0	100
5	128.1
10	144
13	153.5
17	161.2
18	162.6
20	166.3
23	178.9
25	190.2
27	221.8

Figure



this scatter plot could be modeled with either a linear or a cubic

Figure



NORMAL FLOAT AUTO REAL RADIAN MP
CubicReg
y=ax³+bx²+cx+d
a=.0154590051
b=-.5951424724
c=9.150171681
d=98.43272255

Figure