5.2 Properties of Rational Functions

Rational Functions

A rational function has the form

$$R(x) = \frac{p(x)}{q(x)}$$

where $p(x)$ and $q(x)$ are polynomials and $q(x)$ is not the zero polynomial.

* The domain is the set of all real numbers *except* x values that cause the denominator to equal zero.

$q(x) \neq 0$

Ex: Find the domain of the function:

$$f(x) = \frac{x^2 - 5x - 6}{2x^3 - 32x}$$

$2x^3 - 32x \neq 0$

$2x(x^2 - 16) \neq 0$

$2x(x+4)(x-4) \neq 0$

$2x \neq 0 \quad x+4 \neq 0 \quad x-4 \neq 0$

D: $\{x \mid x \neq 0, \pm 4\}$

Asymptotes

Some rational functions contain asymptotes.

An asymptote is a line (or curve) that approaches a given function arbitrarily closely.

There are three types of asymptotes that we will examine:

1) Vertical Asymptotes (can never be crossed)
2) Horizontal Asymptotes (are sometimes crossed)
3) Oblique or Slant Asymptotes (are sometimes crossed)

Vertical Asymptotes

Step 1: Simplify the rational function (if possible).

Step 2: Set the denominator equal to zero.

Ex: $f(x) = \frac{x^2-25}{x^2+7x+12} = \frac{(x+5)(x-5)}{(x+3)(x+4)}$ does not simplify

Vert. Asymp. @ $x=-3$, $x=-4$

Example 1: Finding Vertical Asymptotes

Given the rational function

$$R(x) = \frac{x^2 - 9}{x^2 + 4x - 21} = \frac{(x+3)\cancel{(x-3)}}{(x+7)\cancel{(x-3)}}$$

$$= \frac{x+3}{x+7}$$

a) Find the domain. Write the answer in set notation.

b) Find the vertical asymptote(s).

a) $D: \{x \mid x \neq \underline{-7, 3}\}$

b) Vert. Asymp. @ $x = \underline{-7}$

* there is a hole at $x = 3$

Horizontal Asymptotes

If the degree of the numerator is less than the degree of the denominator, then there is a horizontal asymptote with the equation

$$y = 0$$

Example: $$R(x) = \frac{x-1}{x^2+3x-4}$$

Horizontal Asymptotes

If the degree of the numerator is equal to the degree of the denominator, then there is a horizontal asymptote at:

y = the ratio of the leading coefficients

For example, $$R(x) = \frac{8x^2 - x + 2}{4x^2 - 1}$$

The horizontal asymptote is y = $\frac{8}{4} = 2$.

Example 2:

$$R(x) = \frac{x-1}{x^2+3x-4} = \frac{x-1}{(x-1)(x+4)} = \frac{1}{x+4}$$

a. Find the domain in set notation.
b. Find any holes or vertical asymptotes.
c. Find any horizontal asymptotes.
~~d. Sketch a graph.~~

a) $\{x \mid x \neq -4, 1\}$

b) hole @ $x=1$, vert. asym. @ $x=-4$

c) horiz. asymp. @ $y=0$

Oblique or Slant Asymptotes

If the degree of the numerator is one more than the degree of the denominator, then there is an oblique (or slant) asymptote, and there is NO horizontal asymptote. The oblique asymptote equation is the quotient resulting from polynomial long division.

For example,

$$R(x) = \frac{8x^2 + 26x - 7}{4x + 3}$$

$$R(x) = \frac{2x^3 - 9x^2 + 7x + 6}{2x+1}$$

Refresher on Long Division:

Divide $2x^3 - 9x^2 + 7x + 6$ by $2x + 1$

$$\begin{array}{r|l}
 & x^2 - 5x + 6 \\
2x+1 & 2x^3 - 9x^2 + 7x + 6 \\
 & -(2x^3 + x^2) \\
 & \quad -10x^2 + 7x \\
 & \quad -(-10x^2 - 5x) \\
 & \qquad 12x + 6 \\
 & \qquad -(12x + 6) \\
 & \qquad\qquad 0
\end{array}$$

no remainder

Example 3:

$$R(x) = \frac{8x^2 + 26x - 7}{4x + 3}$$ ← not factorable!

$4x+3=0$
$4x=-3$
$x=-3/4$

a. Find the domain in set notation. $\{x \mid x \neq -3/4\}$

b. Find any holes or vertical asymptotes. V.A. @ $x = -3/4$

c. Find any horizontal asymptotes. none

d. Find any oblique asymptotes. $y = 2x + 5$

~~e. Sketch a graph~~

$$
\begin{array}{r l}
 & \quad 2x + 5 \\
4x+3 & \overline{)\,8x^2 + 26x - 7} \\
 & -(8x^2 + 6x) \\
\hline
 & \qquad\quad 20x - 7 \\
 & \qquad -(20x + 15) \\
\hline
 & \qquad\qquad\quad -22
\end{array}
$$

oblique: $y = 2x + 5$

$(y = mx + b)$

(the remainder does not matter here... it is not part of the linear equation for the oblique asymptote)

Note

A function **cannot** have both a horizontal and oblique asymptote.

A function **can** have both a vertical and horizontal asymptote.

A function **can** have both a vertical and oblique asymptote.

Your Turn

$R(x) = \frac{8x^2+2x-15}{2x+3} = \frac{(2x+3)(4x-5)}{2x+3} = 4x-5$

a. Find the domain in set notation. $\{x \mid x \neq -3/2\}$

b. Find any holes or vertical asymptotes. None!

c. Find any horizontal asymptotes. None!

d. Find any oblique asymptotes.

e. ~~Sketch a graph.~~

$$
\begin{array}{r|l}
 & 4x - 5 \\
2x+3 & 8x^2+2x-15 \\
 & -\ 8x^2+12x \\ \hline
 & -10x-15 \\
 & -10x-15 \\ \hline
\end{array}
$$

oblique: $y=4x-5$

Application:

a) $P(0) = 25$ insects

b) $P(60) = 596.15\ldots \approx 596$ insects

A rare species of insect was discovered in the Amazon Rain Forest. To protect the species, environmentalists declared the insect endangered and transplanted the insect into a protected area. The population P of the insect *t* months after being transplanted is

$$P(t) = \frac{50(1 + 0.5t)}{2 + 0.01t} = \frac{50 + 25t}{2 + .01t}$$

a. How many insects were discovered? (Population at t=0)

b. What will the population be after 5 years? $\rightarrow$ 60 months!

c. Determine the horizontal asymptote of P(t). What is the largest population that the protected area can sustain?

horiz. asym. @ $y = \frac{25}{.01} = \boxed{2500}$

The insect population can reach a max. of 2500 in this protected area.